

LOCALIZATION IN ABELIAN CHERN-SIMONS THEORY

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ABSTRACT. Chern-Simons theory on a closed contact three-manifold is studied when the Lie group for gauge transformations is compact, connected and abelian. A rigorous definition of an abelian Chern-Simons partition function is derived using the Faddeev-Popov gauge fixing method. A *symplectic* abelian Chern-Simons partition function is also derived using the technique of non-abelian localization. This physically identifies the symplectic abelian partition function with the abelian Chern-Simons partition function as topological three-manifold invariants. This study leads to a natural identification of the abelian Reidemeister-Ray-Singer torsion as a specific multiple of the natural unit symplectic volume form on the moduli space of flat abelian connections for the class of Sasakian three-manifolds. The torsion part of the abelian Chern-Simons partition function is computed explicitly in terms of Seifert data for a given Sasakian three-manifold.

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1. INTRODUCTION

The goal of this article is to study the Chern-Simons partition function as a rigorous topological three manifold invariant using heuristic techniques that arise in the physics literature. Our starting point is a heuristically defined partition function, defined as a Feynman path integral, that physically describes the quantum amplitude of a given three manifold X with respect to the Chern-Simons action. From a mathematical perspective the partition function is somewhat mysterious and a rigorous, systematic method for its study, in the sense of *constructive quantum field theory*, is currently lacking. It is remarkable, however, that one can make rigorous mathematical predictions using the partition function.

Our motivation for this study is contained primarily in the work of C. Beasley and E. Witten, [BW05], where the Chern-Simons partition function is studied with respect to a contact structure $H \subset TX$ on a three manifold X . The study of Chern-Simons theory with respect to a contact structure is a novel idea that Beasley and Witten introduce in order to explain some empirical observations of L. Rozansky, [Roz96]. Rozansky studied Chern-Simons theory on Seifert manifolds and observed that the contributions from irreducible flat connections were finite loop exact. He then further observed that this was similar to the behaviour of two dimensional Yang-Mills theory, where similar phenomenon are explained by non-abelian localization [Wit92]. Rozansky explicitly suggested the possibility of studying the Chern-Simons path integral using the method of non-abelian localization.

Our main idea in this article is to study the *abelian* partition function with respect to a contact structure following the ideas of [BW05]. We note that the first systematic work on Chern-Simons theory in the physics literature, including in particular the necessary quantization of its coefficient, may be found in the work of S. Deser, R. Jackiw, G. 't Hooft and S. Templeton [DJT82]. The abelian theory is well known and has been studied from several different perspectives. Recall, $G = \text{U}(1)$ Chern-Simons theory is physically interesting and can be realized as a fundamental building block for a theory of the fractional quantum Hall effect, [Wil82], [FZ91]. Schwarz [Sch79a], [Sch79b] has also shown that the $\text{U}(1)$ theory is related to the Reidemeister-Ray-Singer torsion [Rei35], [RS73], a classical topological invariant of three manifolds. Abelian Chern-Simons theory is also closely related to the one loop contribution of Beasley and Witten's work, which is also a motivation for this study.

Although the Chern-Simons partition function should be a topological invariant *a priori*, the actual process we use to make the partition function rigorous involves a *gauge choice* and requires a choice of metric g on X . By making this gauge choice one obtains a metric dependent quantity that needs to be corrected to recover topological invariance. Once this is done, one may choose any convenient metric to perform calculations and it turns out that the choice of a Sasakian metric greatly facilitates this process. Note that the choice of a Sasakian metric is natural from a physical perspective [GIJP03], and is equivalent to making a "Kaluza-Klein *Ansatz*" for the metric tensor, which is directly related to the dimensional reduction used in this article. A Sasakian three-manifold admits a locally free $\text{U}(1)$ action that acts by isometries for a special metric, g , given by the Sasakian structure on X . This condition is otherwise known as a Killing-contact, or *K-contact*, structure because the vector field generated by the $\text{U}(1)$ action is a Reeb vector field for a contact form $\kappa \in \Omega^1(X, \mathbb{R})$ and is Killing for the metric g . Note that we do not provide an overview of Sasakian or contact geometry in this article and instead we refer to [BG08]. It is interesting to observe that any oriented three-manifold admits a contact structure by a result of Martinet, [Mar71], and in fact our analysis begins by requiring only such a choice. The restriction to a Sasakian structure is made only to simplify our computation. Eventually, we would like to generalize our method for studying the partition function to any three-manifold using the symmetry of a general contact structure.

Recall that Witten has shown [Wit89] that the one loop contribution to the partition function requires some of the work of Atiyah, Patodi and Singer, [APS75a], [APS75b], [APS76],

in order to extract a topological invariant of a three manifold. A generalization of our study will naturally involve making sense of an analogue of some of the work of Atiyah, Patodi, and Singer for three manifolds relative to an arbitrary contact structure, going beyond the Sasakian case.

There are two natural mathematical objects that show up in abelian Chern-Simons theory. The first object is the abelian Reidemeister-Ray-Singer torsion of a closed, orientable three-manifold X , denoted T_X . The Reidemeister-Ray-Singer torsion is a classical three-manifold topological invariant [Rei35], [RS73] and our goal is to prove that its square-root can be identified as a natural symplectic volume form on the moduli space of flat abelian connections in the case that X admits a Sasakian structure. The second mathematical object that shows up naturally in abelian Chern-Simons theory is the eta-invariant for the odd signature operator, L° , acting on $\Omega^1(X, \mathbb{R}) \oplus \Omega^3(X, \mathbb{R})$, which is roughly defined by analytic continuation as a limit,

$$(1) \quad \eta_{\text{grav}}(g) := \lim_{s \rightarrow 0} \sum_{\lambda \in \text{spec}^*(L^\circ)} \text{sgn}(\lambda) |\lambda|^{-s}.$$

In fact, we naturally consider the combination,

$$(2) \quad \frac{\eta_{\text{grav}}(g)}{4} + \frac{1}{12} \frac{\text{CS}(A^g)}{2\pi},$$

where for some trivializing section s ,

$$(3) \quad \text{CS}_s(A^g) := \frac{1}{4\pi} \int_X s^* \text{Tr}(A^g \wedge dA^g + \frac{2}{3} A^g \wedge A^g \wedge A^g),$$

is the gravitational Chern-Simons term. By an Atiyah-Patodi-Singer theorem, [APS75b, Prop. 4.19], the quantity in Eq. (2) is a topological invariant of X depending only on a choice of two-framing. The choice of two-framing is intimately related to the underlying choice of contact structure on X and this will be elaborated upon in future work. Beasley and Witten [BW05] relate the two-framing dependence to a choice of *Seifert framing* on X in the case that the contact structure is of Sasaki type, and we note that one should be able to extend this construction to arbitrary contact structures. This article will focus on the torsion dependent part of the partition function, T_X .

Recall, [BW05] studies the Chern-Simons partition function, [BW05, Eq. 3.1], which is heuristically defined as follows,

$$(4) \quad Z(k) := \frac{1}{\text{Vol}(\mathcal{G})} \left(\frac{k}{4\pi^2} \right)^{\Delta_{\mathcal{G}}} \oint \mathcal{D}A \, e^{ik \text{CS}(A)}.$$

Remark 1. The notation, \oint , is introduced to explicitly distinguish a “path integral” from ordinary integration. The notation, \oint , describes an “integral” over the “space of connections,” and serves as a heuristic device that is generally non-rigorous. Quantities involving integrals over the moduli space of flat connections \mathcal{M}_P are rigorously defined and derived from the former explicitly in this article.

The main ingredients of the path integral of Eq. (4) are described as follows.

- $A \in \mathcal{A}_P = \{A \in (\Omega^1(P) \otimes \mathfrak{g})^G \mid A(\xi^\#) = \xi, \forall \xi \in \mathfrak{g}\}$ is a connection on a principal G -bundle $\pi : P \rightarrow X$ over a closed three-manifold X . In fact, [BW05] consider only G compact, connected, simply connected and simple,
- $\mathfrak{g} = \text{Lie}G$ and $\xi^\# \in \Gamma(TP)$ is the vector field on P generated by the infinitesimal action of $\xi \in \mathfrak{g}$ on P ,
- $k \in \mathbb{Z}$, thought of as an element of $H^4(BG, \mathbb{Z})$ that parameterizes the possible Chern-Simons invariants,
- $\mathcal{G} := \{\psi \in (\text{Diff}(P, P))^G \mid \pi \circ \psi = \pi\}$ is the *gauge group*,
- $\Delta(\mathcal{G})$ is formally defined as the dimension of the gauge group. Note that the definition of the Chern-Simons partition function in Eq. (4) is completely heuristic. The measure $\mathcal{D}A$ has not been defined, but only assumed to “exist heuristically,” and the volume and dimension of the gauge group, $\text{Vol}(\mathcal{G})$ and $\Delta(\mathcal{G})$, respectively, are at best formally defined.
- $\text{CS}(A) := \frac{1}{4\pi} \int_X \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$ is the Chern-Simons action. We abuse all the usual notation here. Strictly speaking, one should write,

$$\text{CS}(A) := \frac{1}{4\pi} \int_X s^* \text{Tr} \left(A \wedge F_A - \frac{1}{6} A \wedge [A \wedge A] \right) \mod (2\pi\mathbb{Z}),$$

where $\text{Tr}(A \wedge A \wedge A) := \frac{1}{2} \text{Tr}(A \wedge [A \wedge A])$, and $s : X \rightarrow P$ is a section. Let $\vartheta \in \Omega^1(G, \mathfrak{g})$ denote the Maurer-Cartan form on G . $\text{Tr} : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{R}$ is defined to be the unique Ad-invariant negative definite quadratic form on \mathfrak{g} normalized for G simply connected so that $\frac{1}{48\pi^2} \text{Tr}(\vartheta \wedge [\vartheta \wedge \vartheta])$ represents an integral class in $H^3(G, \mathbb{R})$. Note that the partition functions of Eq.’s (4) and (5) are defined implicitly with respect the pullback of some trivializing section of the principal G -bundle P . Of course, every principal G -bundle over a three-manifold for G simply connected is trivializable. It is basic fact that the partition functions of Eq.’s (4) and (5) are independent of the choice of such trivializations.

In general, the partition function of Eq. (4) is not known to admit a general mathematical interpretation in terms of the cohomology of some classical moduli space of connections, in contrast to Yang-Mills theory for example, [Wit92]. The main result of [BW05], however, is that if X is assumed to carry the additional geometric structure of a Seifert manifold, then the partition function of Eq. (4) *does* admit a more conventional interpretation in terms of the cohomology of some classical moduli space of connections. Using the additional Seifert structure on X , [BW05] decouple one of the components of a gauge field A , and introduce a “new” partition function denoted by $\bar{Z}(k)$ and given as [BW05, Eq. 3.7],

$$(5) \quad K \cdot \int \mathcal{D}A \mathcal{D}\Phi \exp \left[ik \left(\text{CS}(A) - \frac{1}{4\pi} \int_X 2\kappa \wedge \text{Tr}(\Phi F_A) + \frac{1}{4\pi} \int_X \kappa \wedge d\kappa \text{Tr}(\Phi^2) \right) \right],$$

where

- $K := \frac{1}{\text{Vol}(\mathcal{G}) \text{Vol}(\mathcal{S})} \left(\frac{k}{4\pi^2} \right)^{\Delta \mathcal{G}}$,
- $\kappa \in \Omega^1(X, \mathbb{R})$ is a contact form associated to the Seifert fibration of X , [BW05, §3.2],
- $\Phi \in \Omega^0(X, \mathfrak{g})$ is a Lie algebra-valued zero form on X ,
- $\mathcal{D}\Phi$ is a measure on the space of fields Φ . The measure $\mathcal{D}\Phi$ is defined independently of any metric on X and is formally defined by the positive definite quadratic form

$$(\Phi, \Phi) := - \int_X \kappa \wedge d\kappa \text{Tr}(\Phi^2),$$

which is invariant under the choice of representative for the contact structure (X, H) on X , i.e. under the scaling $\kappa \mapsto f\kappa$, $\Phi \mapsto f^{-1}\Phi$, for some non-zero function $f \in \Omega^0(X, \mathbb{R})$,

- \mathcal{S} is the space of local *shift symmetries*. \mathcal{S} may be identified with $\Omega^0(X, \mathfrak{g})$, and for each $\sigma \in \Omega^0(X, \mathfrak{g})$ the shift symmetry is defined as a local variation $\delta_\sigma : \mathcal{A}_P \rightarrow T\mathcal{A}_P$, where $T\mathcal{A}_P$ denotes the tangent bundle of \mathcal{A}_P . Explicitly, for $A \in \mathcal{A}_P$, $\delta_\sigma(A) := \sigma\kappa \in T_A\mathcal{A}_P$. The shift symmetry is also defined on fields $\Phi \in \Omega^0(X, \mathfrak{g})$ by the variation $\delta_\sigma(\Phi) := \sigma$,
- $F_A \in \Omega^2(X, \mathfrak{g})$ is the curvature of A .

[BW05] then give a heuristic argument showing that the partition function computed using the alternative description of Eq. (5) should be the same as the Chern-Simons partition function of Eq. (4). In essence, they show, [BW05, pg.13],

$$(6) \quad Z(k) = \bar{Z}(k),$$

by gauge fixing $\Phi = 0$ using the shift symmetry. [BW05] then observe that the Φ dependence in the integral can be eliminated by simply performing the Gaussian integral over Φ in Eq. (5) directly. They obtain the alternative formulation,

$$(7) \quad Z(k) = \bar{Z}(k) = K' \cdot \oint \mathcal{D}A \exp \left[ik \left(\text{CS}(A) - \frac{1}{4\pi} \int_X \frac{1}{\kappa \wedge d\kappa} \text{Tr} [(\kappa \wedge F_A)^2] \right) \right],$$

where $K' := \frac{1}{\text{Vol}(\mathcal{G})} \frac{1}{\text{Vol}(\mathcal{S})} \left(\frac{-ik}{4\pi^2} \right)^{\Delta_{\mathcal{G}}/2}$. Note that we follow [BW05, Eq. 3.9] here and abuse notation slightly by writing $\frac{1}{\kappa \wedge d\kappa}$. We have done this with the understanding that since $\kappa \wedge d\kappa$ is non-vanishing (since κ is a contact form), then $\kappa \wedge F_A = \phi \kappa \wedge d\kappa$ for some function $\phi \in \Omega^0(X, \mathfrak{g})$, and we identify $\frac{\kappa \wedge F_A}{\kappa \wedge d\kappa} := \phi$.

The original argument of [BW05] was to decouple one of the components of the gauge field $A \in \mathcal{A}_P$ by introducing a *local shift symmetry* (see [BW05, §3.1]) and then to translate the Chern-Simons partition function into a “moment map squared” form using this symmetry. The general “moment map squared” form for the partition function is a symplectic integral of the canonical form,

$$\bar{Z}(\epsilon) = \frac{1}{\text{Vol}(H)} \left(\frac{1}{2\pi\epsilon} \right)^{\Delta_H/2} \oint_Y \exp \left[\Omega - \frac{1}{2\epsilon} (\mu, \mu) \right],$$

where Y is a symplectic manifold with symplectic form Ω , and H is a Lie group that acts on Y in a Hamiltonian fashion with moment map μ . $\Delta_H = \dim(H)$ and $\epsilon = \frac{2\pi}{k}$. The technique of non-abelian localization [Wit92] can then be applied to study such integrals. This article studies the analogous theory in the case of a compact, connected abelian Lie group G .

The main goal of this article is to study an analogue of the partition function in (5) for the case of a compact, connected and abelian structure group, and to study the equivalence (6) rigorously. In order to study (6) rigorously, we first make $Z(k)$ and $\bar{Z}(k)$ rigorous using completely independent methods and then compare the resulting quantities. Therefore, (6) becomes a rigorous conjecture and this article establishes part of this conjecture by proving that the volume forms that arise in the two cases agree.

In §2 we derive a definition of the abelian partition function, $Z_{\mathbb{T}}(X, P, k)$, which is the

abelian analogue of (4) and is the main topological invariant studied in this article. Our derivation starts with a heuristically defined partition function, as introduced in remark 5,

$$Z_{\mathbb{T}}(X, P, k) = \frac{1}{\text{Vol}(\mathcal{G})} \oint_{\mathcal{A}_P} \mathcal{D}A e^{ik \text{CS}_{X,P}(A_P)},$$

where we abuse notation and write $Z_{\mathbb{T}}(X, P, k)$ for both the heuristic and rigorous versions of the partition function. Our method uses the “Faddeev-Popov gauge fixing method,” as introduced in [FP67], to extract a rigorously defined topological invariant. We note that our method differs from previous derivations [Sch79b], [Man98] of a rigorous candidate for an abelian Chern-Simons partition function. Our approach generalizes more readily to non-abelian gauge groups and also leads to a different and more natural k -dependence for the rigorous quantity we obtain. We also note that our approach differs from that of [Man98] in that we take into account a dependence of the partition function on a choice of two-framing on X . Our final rigorous definition of the abelian Chern-Simons partition function is given in definition 10.

In §3-§7 we study the method of non-abelian localization as introduced by Beasley and Witten [BW05] and apply it to the abelian partition function. First we review some well known localization results in §3, which historically motivated the technique of non-abelian localization originally introduced by Witten [Wit92]. We start by recalling the method of stationary phase, which is a foundational result for subsequent localization ideas. We then briefly review the Duistermaat-Heckman theorem [DH82], and the Berline-Vergne-Atiyah-Bott theorem [BV83], [AB84]. In §4 we review non-abelian localization in general and state and prove some of Witten’s [Wit92] main results in propositions 16 and 17. In §5 we study the “shift symmetry” construction introduced in [BW05], and apply this to the heuristic abelian partition function. The main result of this section is a “new” heuristic definition of what we call the *shift reduced abelian Chern-Simons partition function* given in equation (95). In §6 our main objective is to present the shift reduced abelian Chern-Simons partition function in the canonical moment map squared form,

$$Z(\epsilon) = \frac{1}{\text{Vol}(H)} \left(\frac{1}{2\pi\epsilon} \right)^{\Delta_H/2} \oint_Y \exp \left[\Omega - \frac{1}{2\epsilon}(\mu, \mu) \right],$$

where Y is a symplectic manifold with symplectic form Ω , and H is a Lie group that acts on Y in a Hamiltonian fashion with moment map μ . $\Delta_H = \dim(H)$ and $\epsilon = \frac{2\pi}{k}$. Following the basic argument of [BW05], we are able to obtain this result in equation (107). This allows us to formally apply the method of non-abelian localization to the heuristic path integral in §7 to finally obtain a “new” rigorous definition of the partition function as in definition 23.

In §8 we study the moduli space of flat abelian connections \mathcal{M}_X on a Sasakian three manifold X . The main result in this section provides a fairly concrete description of \mathcal{M}_X and presents its number of components numerically in terms of the natural Seifert data associated to the given Sasakian structure. This result is stated concisely in theorem 27.

In summary, we make two rigorous definitions in the article, the *abelian Chern-Simons partition function* $Z_{\mathbb{T}}(X, k)$ in definition 10 and the *symplectic abelian Chern-Simons partition function* $\bar{Z}_{\mathbb{T}}(X, k)$ in definition 23, using heuristic techniques. These definitions are given as follows.

Definition 2. Let $k \in \mathbb{Z}$ and X a closed, oriented three-manifold. The abelian Chern-Simons partition function, $Z_{\mathbb{T}}(X, k)$, is the quantity,

$$(8) \quad Z_{\mathbb{T}}(X, k) = \frac{1}{|W|} \cdot \sum_{P \in \text{Tors } H^2(X, \Lambda)} Z_{\mathbb{T}}(X, P, k),$$

where $|W|$ is the order of the Weyl group for $\text{SU}(N+1)$ and,

$$(9) \quad Z_{\mathbb{T}}(X, P, k) := k^{m_X} e^{ik \text{CS}_{X,P}(A_P)} e^{\pi i N \left(\frac{\eta_{\text{grav}}(\mathbf{g})}{4} + \frac{1}{12} \frac{\text{CS}(A^{\mathbf{g}})}{2\pi} \right)} \int_{\mathcal{M}_P} \sqrt{T_X},$$

where $m_X = \frac{N}{2}(\dim H^1(X, \mathbb{R}) - 2 \dim H^0(X, \mathbb{R}))$.

Definition 3. [McL12] Let $k \in \mathbb{Z}$, and let $(X, \phi, \xi, \kappa, \mathbf{g})$ be a closed oriented quasi-regular Sasakian three-manifold with associated principal bundle structure,

$$\begin{array}{ccc} \mathbb{U}(1) & \hookrightarrow & X \\ & \downarrow & \\ & \Sigma & \end{array}$$

where Σ is an orbifold such that X has associated Seifert invariants,

$$[g, n; (\alpha_1, \beta_1), \dots, (\alpha_M, \beta_M)],$$

for $\gcd(\alpha_j, \beta_j) = 1$. Define the symplectic abelian Chern-Simons partition function,

$$(10) \quad \bar{Z}_{\mathbb{T}}(X, k) = \frac{1}{|W|} \cdot \sum_{[P] \in \text{Tors } H^2(X, \Lambda)} \bar{Z}_{\mathbb{T}}(X, P, k),$$

where $|W| = (N+1)!$ is the order of the Weyl group for $\text{SU}(N+1)$ and,

$$(11) \quad \bar{Z}_{\mathbb{T}}(X, P, k) = k^{m_X} e^{ik \text{CS}_{X,P}(A_P)} e^{i\pi \left(\frac{N}{4} - \frac{1}{2} \eta_0 \right)} \int_{\mathcal{M}_P} K_X \cdot \omega_P,$$

where, $m_X := \frac{N}{2}(\dim H^1(X; \mathbb{R}) - 2 \dim H^0(X; \mathbb{R}))$, $K_X := \frac{1}{|c_1(X) \cdot \prod_i \alpha_i|^{N/2}}$, $\omega_P := \frac{(\sum_{j=1}^g d\theta_j \wedge d\bar{\theta}_j)^{gN}}{(gN)!(2\pi)^{2gN}}$.

We find that both $Z_{\mathbb{T}}(X, k)$ and $\bar{Z}_{\mathbb{T}}(X, k)$ have identical k -dependent terms k^{m_X} . We note that our k -dependence differs from that in [Man98] since we take into account the k -dependence due to the isotropy group I . Our computation physically identifies the volume form dependent parts of $Z_{\mathbb{T}}(X, k)$ and $\bar{Z}_{\mathbb{T}}(X, k)$, and computes,

$$(12) \quad \sqrt{T_X} = \frac{\omega_P}{|c_1(X) \cdot \prod_i \alpha_i|^{N/2}} = \frac{\omega_P}{\sqrt{|\text{Tors } H^2(X, \Lambda)|}}.$$

Note that previous work also identifies the eta invariant dependent parts of $Z_{\mathbb{T}}(X, k)$ and $\bar{Z}_{\mathbb{T}}(X, k)$. This argument is summarized in [JM10] and uses the main result of [McL10]. We note that the main result of [McL10] assumes a natural choice of *Seifert two-framing* on X and does not study how the partition function changes under a change in two-framing associated to a change in the underlying contact structure. We leave this to future work.

Let $k \in \mathbb{Z}$, and X is a closed oriented three-manifold that admits a quasi-regular Sasakian structure (κ, Φ, ξ, g) , with associated principal bundle structure,

$$\begin{array}{ccc} \mathbb{U}(1) & \hookrightarrow & X \\ & & \downarrow \\ & & \Sigma \end{array}$$

where Σ is an orbifold such that X has associated Seifert invariants,

$$[g, n; (\alpha_1, \beta_1), \dots, (\alpha_M, \beta_M)],$$

for $\gcd(\alpha_j, \beta_j) = 1$. Then this article identifies the magnitudes of $Z_{\mathbb{T}}(X, k)$ and $\bar{Z}_{\mathbb{T}}(X, k)$,

$$(13) \quad |Z_{\mathbb{T}}(X, k)| = |\bar{Z}_{\mathbb{T}}(X, k)|,$$

and finds,

$$(14) \quad |Z_{\mathbb{T}}(X, k)| = \frac{k^{m_X}}{|W|} \cdot \frac{\left| \sum_{[P] \in \text{Tors } H^2(X, \Lambda)} e^{ik \text{CS}_{X,P}(A_P)} \right|}{\sqrt{|\text{Tors } H^2(X, \Lambda)|}},$$

where $m_X := \frac{N}{2}(\dim H^1(X; \mathbb{R}) - 2 \dim H^0(X; \mathbb{R}))$, $|W| = (N+1)!$ is the order of the Weyl group for $\text{SU}(N+1)$ and $\text{Tors } H^2(X, \Lambda)$ is the torsion part of $H^2(X, \Lambda)$ with values in the integral lattice Λ . Lastly, we note that it would be of interest to compute the quantity,

$$(15) \quad \frac{\left| \sum_{[P] \in \text{Tors } H^2(X, \Lambda)} e^{ik \text{CS}_{X,P}(A_P)} \right|}{|W|},$$

explicitly in terms of Seifert data associated to a given quasi-regular Sasakian structure on X , and indeed it would be interesting to make an explicit computation of this quantity on a general closed three-manifold.

2. THE ABELIAN PARTITION FUNCTION

In this section we introduce a partition function, $Z_{\mathbb{T}}(X, k)$, for abelian Chern-Simons theory. A closely related partition function is studied in [Man98], where it shown that it defines a unitary topological quantum field theory as defined by Atiyah in [Ati89]. Our approach is to “derive” a rigorous definition of the partition function from a heuristically defined path integral using the Faddeev-Popov method. Our definition of the abelian Chern-Simons partition function differs from [Man98, Eq. 7.28] in that we take into account a dependence of the partition function on a choice of two-framing on X . We follow [Wit89] and revise the definition of [Man98] by adding a “counterterm,” the gravitational Chern-Simons term, to the eta-invariant that shows up in our considerations. By an Atiyah-Patodi-Singer theorem [APS75b, Prop. 4.19], this counterterm effectively restores topological invariance for the partition function. We note that the existence of such a counterterm is expected from a physics perspective. That is, the abelian Chern-Simons partition function should be a topological invariant *a priori*, since this theory is “generally covariant.” In order to actually *define* the partition function using our methods one needs to make a gauge choice, which in our case is tantamount to a choice of metric g on X . In physics terminology, this choice introduces a quantum anomaly that is made manifest in the dependence of the eta-invariant on the choice of metric g . Introducing the counterterm effectively cancels this anomaly.

Before we define the partition function, we establish some notation and terminology. Let \mathbb{T}

denote a compact, connected abelian Lie group of dimension N , \mathfrak{t} denote its Lie algebra and $\Lambda \subset \mathfrak{t}$ the integral lattice. Let $\text{Tors } H^2(X, \Lambda)$ denote the torsion subgroup of $H^2(X, \Lambda)$. \mathcal{A}_P is the affine space of connections on P modeled on the vector space $\Omega^1(X, \mathfrak{t})$. $\mathcal{G} := \text{Map}(X, \mathbb{T})$ is the group of gauge transformations and acts on \mathcal{A}_P in the standard way. That is, for $g \in \text{Map}(X, \mathbb{T})$, and $A_P \in \mathcal{A}_P$, $A_P \cdot g := A_P + g^* \vartheta$, where $\vartheta \in \Omega^1(\mathbb{T}, \mathfrak{t})$ denotes the Maurer-Cartan form on \mathbb{T} . $\text{CS}_{X,P}(A_P)$ is the Chern-Simons functional of a \mathbb{T} -connection A_P on $P \rightarrow X$ and we describe this presently. For any \mathbb{T} -connection $A_P \in \mathcal{A}_P$, we define an $\text{SU}(N+1)$ -connection \hat{A}_P on an associated principal $\text{SU}(N+1)$ -bundle,

$$(16) \quad \hat{P} = P \times_{\mathbb{T}} \text{SU}(N+1),$$

via,

$$\hat{A}_P|_{[p,h]} = \text{Ad}_{h^{-1}}(\iota_* \text{pr}_1^* A_P|_p) + \text{pr}_2^* \vartheta_h,$$

where $\iota : \mathbb{T} \rightarrow \text{SU}(N+1)$ is inclusion as a maximal torus, $\text{pr}_1 : P \times \text{SU}(N+1) \rightarrow P$ and $\text{pr}_2 : P \times \text{SU}(N+1) \rightarrow \text{SU}(N+1)$ are the standard projections. Since for any three manifold X , \hat{P} is trivializable, let $\hat{s} : X \rightarrow \hat{P}$ be a global section. The definition we use for the Chern-Simons action, $\text{CS}_{X,P}(A_P)$, is as follows,

Definition 4. *The Chern-Simons action functional of a \mathbb{T} -connection $A_P \in \mathcal{A}_P$ is defined by,*

$$(17) \quad \text{CS}_{X,P}(A_P) := \frac{1}{4\pi} \int_X \hat{s}^* \alpha(\hat{A}_P) \mod (2\pi\mathbb{Z}),$$

where $\alpha(\hat{A}_P) \in \Omega^3(\hat{P}, \mathbb{R})$ is the Chern-Simons form of the induced $\text{SU}(N+1)$ -connection $\hat{A}_P \in \mathcal{A}_{\hat{P}}$,

$$(18) \quad \alpha(\hat{A}_P) := \text{Tr}(\hat{A}_P \wedge F_{\hat{A}_P}) - \frac{1}{6} \text{Tr}(\hat{A}_P \wedge [\hat{A}_P, \hat{A}_P]),$$

where $\text{Tr} : \mathfrak{su}(N+1) \otimes \mathfrak{su}(N+1) \rightarrow \mathbb{R}$ denotes the standard Ad -invariant bilinear form in the $(N+1)$ -dimensional representation.

Remark 5. *One may then heuristically define a “partition function” as follows. Let $k \in \mathbb{Z}$ and X a closed, oriented three-manifold. The abelian Chern-Simons partition function, $Z_{\mathbb{T}}(X, k)$, is the heuristic quantity,*

$$(19) \quad Z_{\mathbb{T}}(X, k) = \frac{1}{|W|} \cdot \sum_{P \in \text{Tors } H^2(X, \Lambda)} Z_{\mathbb{T}}(X, P, k),$$

where $|W| = (N+1)!$ is the order of the Weyl group for $\text{SU}(N+1)$ and,

$$(20) \quad Z_{\mathbb{T}}(X, P, k) = \frac{1}{\text{Vol}(\mathcal{G})} \oint_{\mathcal{A}_P} \mathcal{D}A e^{ik \text{CS}_{X,P}(A_P)}.$$

Note that $\text{Vol}(\mathcal{G})$ formally denotes the volume of the gauge group.

In our heuristic definition of the abelian partition in the above remark (5) we sum over flat bundle classes corresponding to elements of $\text{Tors } H^2(X, \Lambda)$ because these stationary points of the Chern-Simons action are the flat connections and these are precisely the bundles that admit flat connections. Note that Eq. (20) is a formal expression, where we heuristically assume the existence of the measure $\mathcal{D}A$. It is precisely the quantity $\frac{\mathcal{D}A}{\text{Vol}(\mathcal{G})}$ in Eq. (20) that is *not* well defined. Our goal in this section is to make definition (5) rigorous using the

Faddeev-Popov method [FP67]. We recall the main ingredients that go into the heuristic definition of the partition function in (5) above. First, the measure $\mathcal{D}A$ is *formally* induced by a choice of metric g on X . Let $\langle \cdot, \cdot \rangle : \mathfrak{t} \otimes \mathfrak{t} \rightarrow \mathbb{R}$ be the bilinear form on \mathbb{T} induced by Tr as in definition 4. Then g defines the Hodge star operator, \star , on the tangent space $T_{A_P}\mathcal{A}_P \simeq \Omega^1(X, \mathfrak{t})$, which in turn induces the \mathcal{G} -invariant Riemannian metric,

$$(21) \quad \langle A, B \rangle_{L^2} := \int_X \langle A \wedge \star B \rangle,$$

on \mathcal{A}_P , for $A, B \in T_{A_P}\mathcal{A}_P \simeq \Omega^1(X, \mathfrak{t})$. Observe that,

$$(22) \quad Z_{\mathbb{T}}(X, P, k) = \frac{e^{ik \text{CS}_{X,P}(A_P)}}{\text{Vol}(\mathcal{G})} \int_{\mathcal{A}_P} \mathcal{D}A \exp \left[\frac{ik}{4\pi} \left(\int_X \langle A \wedge dA \rangle \right) \right],$$

where we rewrite the partition function after identifying $\mathcal{A}_P = A_P + \Omega^1(X, \mathfrak{t})$ for a flat base point A_P in \mathcal{A}_P . We then use the Faddeev-Popov method [FP67] to obtain an exact result. We will gauge fix by choosing a metric g on X and fix the *Lorenz gauge condition*,

$$(23) \quad C(A) = d^\dagger A = 0,$$

where on q -forms $d^\dagger = (-1)^{q+1} \star d \star$ is the adjoint of d defined with respect to the Hodge star \star for the metric g . We introduce the gauge fixing action,

$$(24) \quad S_{\text{gauge}}(A, \Psi, \mathbf{c}, \bar{\mathbf{c}}) = \frac{1}{2\pi} \int_X (\langle d^\dagger A \wedge \Psi \rangle + \langle \bar{\mathbf{c}} d^\dagger d \mathbf{c} \rangle),$$

where $\Psi \in \Omega^3(X, \mathfrak{t})$ is a Lagrange multiplier term that enforces the gauge condition (23), and $\mathbf{c}, \bar{\mathbf{c}}$ are formal anti-commuting Lie algebra valued *ghost fields* that allow one to write the measure fixing determinant of $d^\dagger d$ in exponential form. Let I denote the isotropy subgroup of \mathcal{G} at $A_P \in \mathcal{A}_P$. This is the group of constant maps from X to \mathbb{T} since,

$$(25) \quad \theta \in \text{Lie } \mathcal{G} : A_P \mapsto A_P + d\theta,$$

I.e. $d\theta = 0 \Rightarrow \theta = \text{constant}$ (we assume that X is connected), and hence $I \simeq \mathbb{T}$. Let $\text{Vol } I$ be the volume of the isotropy subgroup with respect to the induced measure on \mathcal{G} ,

$$(26) \quad \text{Vol } I = [\text{Vol } X]^{N/2} = \left[\int_X \star 1 \right]^{N/2}.$$

Eq. (26) follows from the definition of the invariant metric on the group \mathcal{G} that is induced by the inner product on $\text{Lie } \mathcal{G} \simeq \Omega^0(X, \mathfrak{t})$ that comes from g ,

$$(27) \quad G_{\mathcal{G}}(\theta, \phi) := \int_X \langle \theta \wedge \star \phi \rangle,$$

where $\theta, \phi \in \text{Lie } \mathcal{G} \simeq \Omega^0(X, \mathfrak{t})$. Observe that $G_{\mathcal{G}}$ restricted to the space of constant functions is simply a scalar multiple of $\langle \cdot, \cdot \rangle_{L^2}$ at each $\Psi \in \mathcal{G} \simeq \text{Map}(X, \mathbb{T})$,

$$\begin{aligned} G_{\mathcal{G}}(\theta, \phi)|_{\Psi} &= \int_X \langle \theta \wedge \star \phi \rangle, \\ &= \left(\int_X \star 1 \right)^{N/2} \langle \theta, \phi \rangle_{L^2}, \end{aligned}$$

since $\theta, \phi \in \mathfrak{t}$ are constant. We may therefore write $\sqrt{G_{\mathcal{G}}} = (\int_X \star 1)^{N/2}$. If $\sqrt{G_{\mathcal{G}}} D\sigma$ denotes the measure on $I < \mathcal{G}$, then,

$$\begin{aligned}
 \text{Vol } I &= \int_{\mathbb{T}} \sqrt{G_{\mathcal{G}}} D\sigma, \\
 &= \sqrt{G_{\mathcal{G}}}, \text{ setting } \int_I D\sigma = 1, \\
 (28) \quad &= \left[\int_X \star 1 \right]^{N/2}.
 \end{aligned}$$

Observe that in (22) we may integrate the gauge orbit out and write our integral over the quotient space $\mathcal{A}_P/\mathcal{G}$. Note that the metric on \mathcal{A}_P formally descends to a metric on the quotient $\mathcal{A}_P/\mathcal{G}$, and thereby induces a quotient measure that we denote by $\widehat{\mathcal{DA}}$. The integral over of the gauge orbit will contribute a factor of,

$$\frac{\text{Vol } \mathcal{G}}{\text{Vol } I},$$

due to the presence of the isotropy group. We now define $Z_{\mathbb{T}}(X, P, k)$ as,

$$(29) \quad K(A_P, k) \cdot \oint \mathcal{DA} \mathcal{D}\Psi \mathcal{D}\mathbf{c} \mathcal{D}\bar{\mathbf{c}} \exp \left[\frac{ik}{4\pi} \int_X \langle A \wedge dA \rangle + ik S_{\text{gauge}}(A, \Psi, \mathbf{c}, \bar{\mathbf{c}}) \right],$$

where,

$$(30) \quad K(A_P, k) := \frac{e^{ik \text{CS}_{X,P}(A_P)}}{\text{Vol } I} \cdot k^{-\frac{1}{2} \dim H^0(X, \mathfrak{t})},$$

and we have included the factor $k^{-\frac{1}{2} \dim H^0(X, \mathfrak{t})}$ in (30) to take into account the k -dependence that occurs in the volume of the isotropy group as in (28). Let,

$$L : \Omega^{\bullet}(X, \mathfrak{t}) \rightarrow \Omega^{\bullet}(X, \mathfrak{t}),$$

denote the self-adjoint operator defined by,

$$L := \star d + d\star,$$

and let L° denote the operator L restricted to the odd forms, $\Omega^1(X, \mathfrak{t}) \oplus \Omega^3(X, \mathfrak{t})$. Observe that the (A, Ψ) dependent part of the action in (29) may be expressed as,

$$\int_X \langle A \wedge dA + 2d^{\dagger} A \cdot \Psi \rangle = \langle (A, \Psi), L^{\circ}(A, \Psi) \rangle_{L^2}.$$

Overall, (29) leads to the following expression for $Z_{\mathbb{T}}(X, P, k)$,

$$\begin{aligned}
 &K(A_P, k) \cdot \oint \mathcal{DA} \mathcal{D}\Psi \mathcal{D}\mathbf{c} \mathcal{D}\bar{\mathbf{c}} \exp \left[\frac{ik}{4\pi} \langle (A, \Psi), L^{\circ}(A, \Psi) \rangle_{L^2} + \frac{ik}{2\pi} \int_X \langle \bar{\mathbf{c}} d^{\dagger} d \mathbf{c} \rangle \right], \\
 &= K'(A_P, k) \cdot \oint \widehat{\mathcal{DA}} \mathcal{D}\Psi \exp \left[\frac{ik}{4\pi} \langle (A, \Psi), L^{\circ}(A, \Psi) \rangle_{L^2} \right] \det' [d^{\dagger} d],
 \end{aligned}$$

where $\mathbf{c}, \bar{\mathbf{c}}$ have been integrated out to obtain the last line, and \det' denotes a regularized determinant to be defined later. As shown in Lemma 7 below, the determinant \det' defined in (49) satisfies the scaling,

$$(31) \quad \det' [c \cdot d^{\dagger} d] = c^{-\dim H^0(X, \mathfrak{t})} \cdot \det' [d^{\dagger} d],$$

for $c \in \mathbb{R}$. We have therefore multiplied $K(A_P, k)$ by the factor,

$$(32) \quad k^{-\dim H^0(X, \mathfrak{t})},$$

and we have,

$$(33) \quad K'(A_P, k) := \frac{e^{ik \text{CS}_{X,P}(A_P)}}{\text{Vol } I} k^{-\frac{3}{2} \dim H^0(X, \mathfrak{t})}.$$

We may define the integral,

$$(34) \quad Z_{\mathbb{T}}(X, P, k) = K'(A_P, k) \cdot \oint \widehat{\mathcal{D}A} \mathcal{D}\Psi \exp \left[\frac{ik}{4\pi} \langle (A, \Psi), L^\circ(A, \Psi) \rangle_{L^2} \right] \det' [d^\dagger d],$$

formally using Gaussian integration. Let $\text{spec}^*(L^\circ)$ denote the non-zero part of the spectrum of L° and formally define the signature of L° ,

$$\text{sgn}(L^\circ) := \sum_{\lambda \in \text{spec}^*(L^\circ)} \text{sign } \lambda,$$

where $\text{sign } \lambda = \pm 1$ denotes the sign of the real number λ . Of course, this expression for the signature of L° is not generally well defined and we will regularize using an eta-invariant to obtain something sensible later. For now, we formally apply Gaussian integration to obtain a rigorously defined quantity. Observe that the stationary points of the action in (34) satisfy,

$$(35) \quad dA = d^\dagger \Psi_0,$$

and,

$$(36) \quad d^\dagger A = 0,$$

where $\Psi_0 \in \Omega^3(X, \mathfrak{t})$ is a non-zero scalar multiple of Ψ . These equations imply that A and Ψ are harmonic and the critical submanifold is $\mathcal{M}_P \times \mathfrak{t}$ where \mathcal{M}_P denotes the moduli space of flat abelian connections on P . We will choose a normalization such that,

$$(37) \quad \int_{\mathfrak{t}} \widehat{\mathcal{D}\Psi} = 1,$$

where $\widehat{\mathcal{D}\Psi}$ is measure induced by $\mathcal{D}\Psi$. Thus, applying Gaussian integration/stationary phase, as in Theorem 11, we obtain,

$$(38) \quad Z_{\mathbb{T}}(X, P, k) = K'(A_P, k) \cdot \int_{\mathcal{M}_P} \frac{1}{\sqrt{|\det' k L^\circ|}} \exp \left[\frac{i\pi}{4} \text{sgn}(L^\circ) \right] \det' [d^\dagger d] \nu,$$

where ν denotes the natural measure on $\mathcal{M}_P \simeq H^1(X, \mathfrak{t})/H^1(X, \Lambda)$, and $\det' L^\circ$ is formally the product of non-zero eigenvalues of L° . We will define $\det' L^\circ$ via regularization using a zeta function determinant. First, we regularize the signature $\text{sgn}(L^\circ)$ via the eta-invariant and set $\text{sgn}(L^\circ) \rightsquigarrow \eta(L^\circ) := \eta(L^\circ)(0)$ where,

$$(39) \quad \eta(L^\circ)(s) := \sum_{\lambda \in \text{spec}^*(L^\circ)} (\text{sgn } \lambda) |\lambda|^{-s}.$$

$\eta(L^\circ)$ has a rigorous mathematical meaning using the fact that $\eta(L^\circ)(s)$ admits a meromorphic extension to \mathbb{C} that is regular at 0, [APS75a].

Remark 6. *The eta-invariant is an analytic invariant introduced by Atiyah, Patodi and Singer [APS75a] defined for an elliptic and self-adjoint operator. As in [APS75a, Prop. 4.20], we may remove some spectral symmetry and the eta invariant of L° coincides with the eta invariant of the operator $\star d$ restricted to $\Omega^1(X, \mathfrak{t}) \cap \text{Im}(d\star)$. Throughout, we will abuse notation slightly and write,*

$$(40) \quad \eta(\star d) = \lim_{s \rightarrow 0} \sum_{\lambda \in \text{spec}^*(\star d)} \text{sgn}(\lambda) |\lambda|^{-s},$$

and replace L° in the notation with $\star d$. We also recall that the expression for the sum,

$$(41) \quad \sum_{\lambda \in \text{spec}^*(\star d)} \text{sgn}(\lambda) |\lambda|^{-s},$$

is defined for large $\text{Re}(s)$ and [APS75a] shows that it has a meromorphic continuation to \mathbb{C} that is analytic at 0. It therefore makes sense to take the limit as $s \rightarrow 0$ in Eq. (40) and to define the eta-invariant $\eta(\star d)$ as evaluation of this limit.

Let $\eta_{\text{grav}}(g)$ be the eta-invariant for the operator $\star d$ acting on $\Omega^1(X, \mathbb{R})$, so that,

$$(42) \quad \eta(\star d) = N \cdot \eta_{\text{grav}}(g),$$

where the eta invariant on the left hand side of (42) is defined on $\Omega^1(X, \mathfrak{t})$ and $N = \dim \mathbb{T}$. Since $\eta(\star d)$ itself is not a topological invariant, we follow [Wit89] and add a “counterterm” that cancels the metric dependence of the eta-invariant. Define,

$$(43) \quad \text{CS}_s(A^g) := \frac{1}{4\pi} \int_X s^* \text{Tr}(A^g \wedge dA^g + \frac{2}{3} A^g \wedge A^g \wedge A^g),$$

the gravitational Chern-Simons term with A^g the Levi-Civita connection and s a trivializing section of twice the tangent bundle of X . More explicitly, let $H = \text{Spin}(6)$, $Q = TX \oplus TX$ viewed as a principal $\text{Spin}(6)$ -bundle over X , $g \in \Gamma(S^2(T^*X))$ a Riemannian metric on X , $\phi : Q \rightarrow \text{SO}(X)$ a principal bundle morphism, and $A^{LC} \in \mathcal{A}_{\text{SO}(X)} := \{A \in (\Omega^1(\text{SO}(X)) \otimes \mathfrak{so}(3))^{\text{SO}(3)} \mid A(\xi^\sharp) = \xi, \forall \xi \in \mathfrak{so}(3)\}$ the Levi-Civita connection. Then $A^g := \phi^* A^{LC} \in \mathcal{A}_Q := \{A \in (\Omega^1(Q) \otimes \mathfrak{h})^H \mid A(\xi^\sharp) = \xi, \forall \xi \in \mathfrak{h}\}$. An Atiyah-Patodi-Singer theorem, [APS75b, Prop. 4.19], says that the combination,

$$(44) \quad \eta_{\text{grav}}(g) + \frac{1}{3} \frac{\text{CS}(A^g)}{2\pi},$$

is a topological invariant depending only on a 2-framing of X . Recall that a 2-framing is a choice of a homotopy equivalence class Π of trivializations of $TX \oplus TX$, twice the tangent bundle of X . Note that Π is represented by the trivializing section $s : X \rightarrow Q$ above. The possible 2-framings correspond to \mathbb{Z} . The identification with \mathbb{Z} is given by the signature defect defined by,

$$\delta(X, \Pi) = \text{sign}(M) - \frac{1}{6} p_1(2TM, \Pi),$$

where M is a 4-manifold with boundary X and $p_1(2TM, \Pi)$ is the relative Pontrjagin number associated to the framing Π of the bundle $TX \oplus TX$. The canonical 2-framing Π^c corresponds to $\delta(X, \Pi^c) = 0$. Thus, overall we replace $\text{sgn } L^\circ$ in (38) with,

$$(45) \quad N \cdot \left[\eta_{\text{grav}}(g) + \frac{1}{3} \frac{\text{CS}(A^g)}{2\pi} \right],$$

which is a topological invariant up to a choice of two-framing on X .

Next, we consider the determinant $\det' L^\circ$ in (38). Recall the Hodge-de Rham Laplacian,

$$(46) \quad \Delta_q := d^\dagger d + dd^\dagger, \text{ on } \Omega^q(X, \mathfrak{t}),$$

Let $\zeta_q(s)$ denote the zeta function of Δ_q ,

$$(47) \quad \zeta_q(s) = \zeta(\Delta_q)(s) := \sum_{\lambda \in \text{spec}^*(\Delta_q)} \lambda^{-s}.$$

Recall, $\zeta_q(s)$ is defined for $\text{Re}(s) \gg 0$ by,

$$(48) \quad \zeta_q(s) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{tr}(e^{t\Delta_q} - \Pi_q) dt,$$

and then analytically continued to \mathbb{C} as usual. Note that $\Pi_q : \Omega^q(M, \rho) \rightarrow \mathcal{H}^q(M, \rho)$ is orthogonal projection, and $\Gamma(s)$ is the gamma function,

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt.$$

The notation \det' refers to a regularized determinant and is defined for the Laplacians Δ_q as,

$$(49) \quad \det'(\Delta_q) := e^{-\zeta'(\Delta_q)(0)}.$$

The scaling used in (31) is a consequence of the following,

Lemma 7. *For any $c \in \mathbb{R}$,*

$$\begin{aligned} \det'[c \cdot \Delta_q] &= c^{\zeta_q(0)} \cdot \det'[\Delta_q], \\ &= c^{-\dim H^q(X, \mathfrak{t})} \cdot \det'[\Delta_q]. \end{aligned}$$

Proof. By definition, $\zeta(c\Delta_q)(s) = c^{-s}\zeta(\Delta_q)(s)$. Taking the derivative of $c^{-s}\zeta(\Delta_q)(s)$ with respect to s and evaluating at $s = 0$ and using the definition (49) yields (50). In order to obtain the precise scaling in (50) we use the following [Mül78],

$$\zeta_q(0) = -\dim \text{Ker } \Delta_q = -\dim H^q(X, \mathfrak{t}).$$

This completes the proof. Q.E.D.

Now we define the determinant $\det' L^\circ$ as,

$$\begin{aligned} \det' L^\circ &:= [\det'(L^\circ)^2]^{1/2}, \\ &= [\det'(\Delta_1 \oplus \Delta_3)]^{1/2}, \\ &= [\det' \Delta_1]^{1/2} \cdot [\det' \Delta_3]^{1/2}. \end{aligned}$$

Note that $\Delta_1 \oplus \Delta_3$ denotes the operator acting on $\Omega^1(X, \mathfrak{t}) \oplus \Omega^3(X, \mathfrak{t})$ in the obvious way, preserving the direct sum. The quantity of interest in equation (38) is,

$$\begin{aligned} \frac{\det'[d^\dagger d]}{\sqrt{|\det'[k L^\circ]|}} &= \frac{\det' \Delta_0}{\sqrt{|\det' k^2 \Delta_1|^{1/2} \cdot [\det' k^2 \Delta_3]^{1/2}|}}, \\ &= k^{\frac{1}{2}(\dim H^1(X, \mathfrak{t}) + \dim H^0(X, \mathfrak{t}))} \cdot \frac{[\det' \Delta_0]^{3/4}}{[\det' \Delta_1]^{1/4}}, \end{aligned}$$

where the last line follows from Lemma 7 and the fact that $\det' \Delta_0 = \det' \Delta_3$ by duality. Overall, we obtain the following for $Z_{\mathbb{T}}(X, P, k)$,

$$(50) \quad K'(A_P, k) \cdot \int_{\mathcal{M}_P} \frac{1}{\sqrt{|\det' k L^\circ|}} \exp \left[\frac{i\pi}{4} \operatorname{sgn}(L^\circ) \right] \det' [d^\dagger d] \nu,$$

$$(51) \quad := k^{m_X} e^{ik \operatorname{CS}_{X,P}(A_P)} e^{\left[\frac{i\pi N}{4} \cdot \left[\eta_{\operatorname{grav}}(g) + \frac{1}{3} \frac{\operatorname{CS}(A^g)}{2\pi} \right] \right]} \int_{\mathcal{M}_P} \frac{1}{\operatorname{Vol} I} \frac{[\det' \Delta_0]^{3/4}}{[\det' \Delta_1]^{1/4}} \nu,$$

where $m_X := \frac{1}{2} (\dim H^1(X, \mathfrak{t}) - 2 \dim H^0(X, \mathfrak{t}))$.

Remark 8. Note that the term $m_X := \frac{1}{2} (\dim H^1(X, \mathfrak{t}) - 2 \dim H^0(X, \mathfrak{t}))$ results in a difference in the k -dependence of our partition function from that of [Man98] by a factor of $k^{-\frac{1}{2} \dim H^0(X, \mathfrak{t})}$. The relevant k -dependent term in [Man98] is $m'_X := \frac{1}{2} (\dim H^1(X, \mathfrak{t}) - \dim H^0(X, \mathfrak{t}))$, and this difference arises because we take into account the contribution to the k -dependence coming from the volume of the isotropy group, whereas [Man98] does not. We also note that we obtain the same k -dependent term k^{m_X} in the symplectic abelian Chern-Simons partition function given in definition 23 using the completely different technique of non-abelian localization.

Next, we will show that the quantity inside the integral in (51),

$$\frac{1}{\operatorname{Vol} I} \frac{[\det' \Delta_0]^{3/4}}{[\det' \Delta_1]^{1/4}} \nu,$$

is precisely the square-root of the Reidemeister-Ray-Singer torsion of X . The Reidemeister-Ray-Singer torsion T_X will be defined as a density on the determinant line,

$$(52) \quad |\det H^\bullet(X, \mathfrak{t})|^* := \bigotimes_{j=0}^3 (\det H^j(X, \mathfrak{t}))^{(-1)^j}.$$

We make the natural identification,

$$(53) \quad H^\bullet(X, \mathfrak{t}) \simeq \mathcal{H}^\bullet(X, \mathfrak{t}),$$

under the de Rham map where $\mathcal{H}^\bullet(X, \mathfrak{t})$ denotes the harmonic forms on X with respect to the Laplacian given in Eq. (46). Let $\delta_{|\det H^\bullet(X, \mathfrak{t})|^*}$ denote the induced density on $|\det H^\bullet(X, \mathfrak{t})|^*$ corresponding to the induced metric from the L^2 metric on $\mathcal{H}^\bullet(X, \mathfrak{t})$. Now make the following,

Definition 9. [RS73] Given a closed Riemannian three manifold (X, g) , define the scalar Reidemeister-Ray-Singer torsion,

$$(54) \quad T_X^{\operatorname{scal}}(g) := \exp \left(\frac{1}{2} \sum_{q=0}^3 (-1)^q q \zeta'(\Delta_q)(0) \right).$$

Define the Reidemeister-Ray-Singer torsion T_X as,

$$(55) \quad T_X := T_X^{\operatorname{scal}}(g) \cdot \delta_{|\det H^\bullet(X, \mathfrak{t})|^*}.$$

Note that $T_X^{\operatorname{scal}}(g)$ is generally dependent upon the choice of metric g and it is shown in [RS73] that T_X is indeed independent of g . Note that given an orthonormal basis for

$\mathcal{H}^q(X, \mathbb{R})$, $\{\nu_1^{[q]}, \dots, \nu_{b_q}^{[q]}\}$, where $b_q := \dim H^1(X, \mathbb{R})$, $\delta_{|\det H^\bullet(X, \mathbb{t})|^*}$ may be written as,

$$(56) \quad \delta_{|\det H^\bullet(X, \mathbb{t})|^*} = \bigotimes_{q=0}^3 |\nu^{[q]}|^{N \cdot (-1)^q},$$

where $\nu^{[q]} := \nu_1^{[q]} \wedge \dots \wedge \nu_{b_q}^{[q]}$ and $N = \dim \mathbb{T}$. Observe that an orthonormal basis for $\mathcal{H}^q(X, \mathbb{R}) = \mathbb{R}$ is a constant $\nu^{[0]}$ such that $|\nu^{[0]}|^N = (\text{Vol } I)^{-1}$. One may see this by computing,

$$\begin{aligned} 1 &= \|\nu^{[0]}\|_{L^2}^2, \\ &= \int_X \nu^{[0]} \wedge \star \nu^{[0]}, \\ &= |\nu^{[0]}|^2 \int_X \star 1. \end{aligned}$$

Combining this with (28) one obtains $|\nu^{[0]}|^N = (\text{Vol } I)^{-1}$. Using Poincaré duality $H^q(X, \mathbb{R}) \simeq H^{3-q}(X, \mathbb{R})^*$ we may write the square-root of $\delta_{|\det H^\bullet(X, \mathbb{t})|^*}$ in (56) as,

$$(57) \quad \begin{aligned} [\delta_{|\det H^\bullet(X, \mathbb{t})|^*}]^{1/2} &= |\nu^{[0]}|^N \otimes |\nu^{[1]}|^N, \\ &= \frac{1}{\text{Vol } I} \cdot \nu, \end{aligned}$$

where we define $\nu := |\nu^{[1]}|^N$. Thus, using Poincaré duality combined with the duality $\Delta_q \simeq \Delta_{3-q}$ induced by the Hodge star, then the definition of T_X in (54), (55) and equation (57) imply that the square-root of the Reidemeister-Ray-Singer torsion can be expressed as,

$$(58) \quad \sqrt{T_X} = \frac{1}{\text{Vol } I} \frac{[\det' \Delta_0]^{3/4}}{[\det' \Delta_1]^{1/4}} \nu,$$

as claimed. Overall, we make the following,

Definition 10. Let $k \in \mathbb{Z}$ and X a closed, oriented three-manifold. The abelian Chern-Simons partition function, $Z_{\mathbb{T}}(X, k)$, is the quantity,

$$(59) \quad Z_{\mathbb{T}}(X, k) = \frac{1}{|W|} \cdot \sum_{P \in \text{Tors } H^2(X, \Lambda)} Z_{\mathbb{T}}(X, P, k),$$

where $|W|$ is the order of the Weyl group for $\text{SU}(N+1)$ and,

$$(60) \quad Z_{\mathbb{T}}(X, P, k) := k^{m_X} e^{ik \text{CS}_{X,P}(A_P)} e^{\pi i N \left(\frac{\eta_{\text{grav}}(\mathbf{g})}{4} + \frac{1}{12} \frac{\text{CS}(A^{\mathbf{g}})}{2\pi} \right)} \int_{\mathcal{M}_P} \sqrt{T_X},$$

where $m_X = \frac{N}{2}(\dim H^1(X, \mathbb{R}) - 2 \dim H^0(X, \mathbb{R}))$.

Note that we can either choose the canonical framing [Ati90] and work with this throughout, or we can observe that if the framing of X is twisted by F units, then $\text{CS}(A^{\mathbf{g}})$ transforms by,

$$\text{CS}(A^{\mathbf{g}}) \rightarrow \text{CS}(A^{\mathbf{g}}) + 2\pi F.$$

The partition function $Z_{\mathbb{T}}(X, k)$ is then transformed by,

$$(61) \quad Z_{\mathbb{T}}(X, k) \rightarrow Z_{\mathbb{T}}(X, k) \cdot \exp\left(\frac{2\pi i N F}{24}\right).$$

Thus, $Z_{\mathbb{T}}(X, k)$ is a topological invariant of framed, oriented three-manifolds, with a transformation law under change of framing. This is tantamount to a topological invariant of oriented three-manifolds without a choice of framing.

We note that it is more natural for us to take the latter perspective, which allows the framing to change with the given transformation law. The reason for this is that the canonical framing is not the most natural choice for our geometric situation and there should be a non-trivial dependence of the framing on the underlying choice of contact structure.

3. BACKGROUND ON NON-ABELIAN LOCALIZATION

In this section we review the general technique of non-abelian localization. First we review *abelian* localization, which serves historically as a motivation for non-abelian localization. A good starting point is the classical stationary phase formula, which is given in the following,

Theorem 11. [GS77, §1, page 6] *Let M be a smooth m -dimensional manifold, ν a smooth compactly supported density of unit weight and $f \in C^\infty(M, \mathbb{R})$ a Morse function (i.e. with non-degenerate Hessian, Hess , on $\text{Crit } f$, the critical points of f), and let $p_l \in \text{Crit } f \cap \text{supp } \nu$, $1 \leq l \leq L$. Then,*

$$(62) \quad \int_M e^{if/\hbar} \nu = (2\pi\hbar)^{m/2} \sum_{l=1}^L \frac{e^{i\pi f(p_l)/\hbar} e^{i \text{sgn } \text{Hess}_{p_l} f/4}}{|\det_\nu \text{Hess}_{p_l} f|^{1/2}} + O(\hbar^{1+m/2}),$$

where $\hbar \in \mathbb{R}^+$, $i = \sqrt{-1}$, $\text{sgn } \text{Hess}_{p_l} f$ denotes the signature of the Hessian, and $|\det_\nu \text{Hess}_{p_l} f|^{1/2} \nu = |\det \langle \text{Hess}_{p_l} f \cdot, \cdot \rangle|^{1/2}$.

The beautiful work of Duistermaat and Heckman [DH82], [DH83] studies Theorem (11) when M is symplectic, the function f is proportional to a moment map and the density is given by the Liouville volume. In this case Duistermaat and Heckman are able to show that the error term in (62) vanishes. Let us set some notation.

- Let M be a compact symplectic manifold with symplectic form $\omega \in \Omega^2(M, \mathbb{R})$, G a (compact) Lie group with Hamiltonian action $G \curvearrowright M$ and moment map $\mu : M \rightarrow \mathfrak{g}^*$. Let \mathbb{T} denote a compact torus Lie group.
- Recall that the moment map $\mu : M \rightarrow \mathfrak{g}^*$ satisfies: $\forall Y \in \mathfrak{g}$, $d\mu^Y = \iota_{Y^\#} \omega$, where $\mu^Y(x) := \langle \mu(x), Y \rangle$ for the dual pairing on \mathfrak{g} , $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \otimes \mathfrak{g} \rightarrow \mathbb{R}$ and $Y^\#(x) := \frac{d}{dt} \big|_{t=0} (x \cdot \exp(tY))$,
- μ is G -equivariant with respect to coadjoint action on \mathfrak{g}^* .
- For $Y \in \mathfrak{g}$, define the fixed point set $T_Y := \overline{\langle \exp(\mathbb{R}Y) \rangle} < G$ compact torus, and $M^Y := \{p \in M \mid T_Y \cdot p = p\} = \{p \in M \mid Y^\#(p) = 0\}$, and assume that it is non-empty with isolated points.
- Let $p \in M^Y$ be an isolated fixed point, and define $\mathfrak{L}_p : \text{Lie}(T_Y) \times T_p M \rightarrow T_p M$ the infinitesimal *isotropy representation*,

$$\mathfrak{L}_p(Y, v_p) := (\mathcal{L}_{Y^\#} v)_p,$$

for any $v \in \Gamma(TM)$ such that $v|_p = v_p$. Note that $\mathfrak{L}_p(Y) := \mathfrak{L}_p(Y, \cdot)$, is invertible $\forall Y \in \text{Lie}(T_Y)$ since $p \in M^Y$ isolated.

The Duistermaat-Heckman theorem may be stated as follows,

Theorem 12. [DH82] *Let $(M, \mathbb{T}, \omega, \mu)$ be a compact, smooth $m = 2n$ -dimensional Hamiltonian \mathbb{T} space with Liouville form $\beta := \omega^n/n!$, and let $Y \in \text{Lie } \mathbb{T}$ such that $M^Y = \{p_l\}_{l=1}^L$ consists of isolated points. Then for any $\hbar \in \mathbb{R}^+$,*

$$(63) \quad \int_M e^{i\mu^Y/\hbar} \beta = (2\pi\hbar)^n \sum_{p_l \in M^Y} \frac{e^{i\pi\mu^Y(p_l)/\hbar} e^{i \text{sgn Hess}_{p_l} \mu^Y/4}}{|\det_\beta \text{Hess}_{p_l} \mu^Y|^{1/2}},$$

$$(64) \quad = (2\pi i \hbar)^n \sum_{p_l \in M^Y} \frac{e^{i\pi\mu^Y(p_l)/\hbar}}{\text{Pf}(\mathfrak{L}_{p_l}(Y))},$$

where $\text{Pf}(\mathfrak{L}_{p_l}(Y))$ denotes the Pfaffian of $\mathfrak{L}_{p_l}(Y)$.

The Duistermaat-Heckman theorem was subsequently generalized by Berline-Vergne [BV83] and Atiyah-Bott [AB84] to more general manifolds and group actions. These results use the notion of equivariant forms and we establish notation presently.

- Let $G \curvearrowright M$ be a compact manifold with action of a compact Lie group G .
- The equivariant differential forms on $G \curvearrowright M$ are $\Omega_G^{\infty, \bullet}(M) := \{\alpha : \mathfrak{g} \rightarrow \Omega^\bullet(M) : g \cdot \alpha(Y) = \alpha(\text{Ad}(g)Y), Y \in \mathfrak{g}, g \in G\}$,
- The equivariant differential is $d_{\mathfrak{g}} : \Omega_G^{\infty, \bullet}(M) \rightarrow \Omega_G^{\infty, \bullet}(M)$ and is defined by,

$$(65) \quad d_{\mathfrak{g}}\alpha(Y) := d\alpha(Y) - \iota_Y \# \alpha(Y),$$

- Recall the natural grading on $\Omega_G^\bullet(M) := (\mathbb{C}[\mathfrak{g}] \otimes \Omega^\bullet(M))^G \subset \Omega_G^{\infty, \bullet}(M)$, the subalgebra of polynomial maps. The degree of $\alpha \in \Omega_G^\bullet(M)$ is defined to be (differential form degree) + 2(polynomial degree).
- Let \mathcal{N} denote the normal bundle of M^Y , which is orientable with even dimensional fibers.

One then has the following,

Theorem 13. (Berline-Vergne-Atiyah-Bott) *Let $G \curvearrowright M$ be a compact, smooth, oriented manifold with action of a compact Lie group G and let $\alpha \in \text{Map}^\infty(\mathfrak{g}, \Omega^\bullet(M))$ be an equivariantly closed differential form on M , i.e. $d_{\mathfrak{g}}\alpha = 0$ (α not necessarily in $\Omega_G^{\infty, \bullet}(M)$). Let $Y \in \mathfrak{g}$ such that M^Y is isolated, then,*

$$(66) \quad \int_M \alpha(Y) = (-2\pi)^{m/2} \sum_{p_l \in M^Y} \frac{\alpha(Y)_{[0]}(p_l)}{\text{Pf}(\mathfrak{L}_{p_l}(Y))},$$

where $\alpha(Y)_{[0]}(p_l)$ denotes the zeroth degree part of α evaluated at p_l .

Of course, Theorem (12) and Theorem (13) apply only when the critical point set M^Y consists of isolated points. This theorem has been generalized to the case of a not necessarily isolated critical point set in the following,

Theorem 14. [BGV92, Theorem 7.13] *Let $G \curvearrowright M$ be a compact, smooth, oriented manifold with action of a compact Lie group G and let $\alpha \in \text{Map}^\infty(\mathfrak{g}, \Omega^\bullet(M))$ be an equivariantly closed differential form on M , i.e. $d_{\mathfrak{g}}\alpha = 0$ (α not necessarily in $\Omega_G^{\infty, \bullet}(M)$). Let $Y \in \mathfrak{g}$ and M^Y denote the set of zeroes of $Y^\#$ in M , and let \mathcal{N} denote the normal bundle of M^Y in M . Then for $\tilde{Y} \in \mathfrak{g}_Y := \{\tilde{Y} \in \mathfrak{g} \mid [\tilde{Y}, Y] = 0\}$ sufficiently close to Y , we have,*

$$(67) \quad \int_M \alpha(\tilde{Y}) = \int_{M^Y} (-2\pi)^{\text{rank}(\mathcal{N})/2} \frac{\alpha(Y)}{\chi_{\mathfrak{g}_Y}^{\mathcal{N}}(\tilde{Y})},$$

where $\chi_{\mathfrak{g}^Y}^{\mathcal{N}}(\tilde{Y})$ is the equivariant Euler form of the normal bundle and $\text{rank}(\mathcal{N}) : M^Y \rightarrow \mathbb{N}_0$ is the locally constant function giving the codimension of the connected components of M^Y .

Note that the Duistermaat-Heckman theorem (12) follows as a corollary of Theorem (13) or Theorem (14) by setting $\alpha(Y) := (2\pi i)^{-n} \exp[i(\omega + \mu^Y/\hbar)] \in \Omega_G^\bullet(M)$.

4. GENERAL NON-ABELIAN LOCALIZATION

This article is concerned with the study of path integrals, which may be viewed as a formal generalization of integrals of the form,

$$\int_M e^{if/\hbar} \nu,$$

as in (62), to infinite dimensions. In the case of interest, the manifold M is replaced by a space of connections \mathcal{A} , f becomes an action functional and ν gets identified as a path integral “measure” $\mathcal{D}A$. What is lost in rigour by taking this perspective is certainly supplemented by the insight one obtains into techniques of calculation of what physicists call “observables” in quantum field theories. There is a vast array of literature on this subject and we will not attempt to provide references here. Instead, we will focus on some work that captures the spirit of the heuristic path integral method and that also has the added virtue of being directly related to our work. A good starting point here is Yang-Mills theory in two dimensions and in particular the work of E. Witten on this subject [Wit92] that introduces the technique of non-abelian localization. Let us establish some notation.

- Let Σ be a compact surface, $P \simeq \Sigma \times G$ a (trivial) principal G bundle over Σ . G a compact, connected, simply connected and simple Lie group, (i.e. $G = \text{SU}(N+1)$, $N \in \mathbb{N}_0$),
- $\mathcal{A} = \{A \in (\Omega^1(P) \otimes \mathfrak{g})^G \mid A(\xi^\#) = \xi, \forall \xi \in \mathfrak{g}\}$ is the space of connections,
- $\mathcal{G} := (C^\infty(P, G))^G$ is the *gauge group*,
- $\omega \in \Omega^2(\mathcal{A})$ is defined for $\eta, \chi \in \Omega^1(\Sigma) \otimes \mathfrak{g} = T_{A_0}(\mathcal{A})$, as $\omega_{A_0}(\eta, \chi) := -\int_\Sigma \text{Tr}(\eta \wedge \chi)$, “Tr” denotes a normalized Ad-invariant, negative-definite quadratic form on \mathfrak{g} (i.e. normalized Killing form on $\mathfrak{su}(N+1)$). One can show that ω is a symplectic form on \mathcal{A} ,
- $\mathcal{G} \curvearrowright \mathcal{A}$ is the gauge group action defined by, $A \cdot u := \text{Ad}_{u^{-1}} A + u^* \theta$, where θ is the Maurer-Cartan form on G . One calls the action of gauge group elements, “gauge transformations.”

Define $\mu : \mathcal{A} \rightarrow \text{Lie}(\mathcal{G})^*$ via,

$$\mu(A) := F_A = dA + \frac{1}{2}[A \wedge A] \in (\Omega^2(P) \otimes \mathfrak{g})^G,$$

the curvature of A . Then Atiyah-Bott show [AB83] that $(\mathcal{A}, \omega, \mathcal{G}, \mu)$ is a Hamiltonian \mathcal{G} -space. This is the basic observation that will allow us to set up the analogy between the Duistermaat-Heckman theorem (12) and the computation of the two dimensional Yang-Mills partition function using non-abelian localization. Next we recall the *action* functional for the Yang-Mills partition function, which formally replaces the function f in (62). The action functional for two dimensional Yang-Mills theory is defined as,

$$(68) \quad S_{YM}[A] := -\frac{1}{2} \int_\Sigma \text{Tr}(F_A \wedge \star F_A).$$

One can show that S_{YM} is invariant under gauge transformations. The Yang-Mills partition function, which formally replaces the integral in (62), is defined as,

$$(69) \quad Z(\hbar) := \frac{1}{\text{Vol}(\mathcal{G})} \left(\frac{-i}{2\pi\hbar} \right)^{\Delta_{\mathcal{G}}/2} (-i)^{\Delta_{\mathcal{A}}/2} \oint_{\mathcal{A}} \exp \left[i \left(\omega + \frac{1}{\hbar} S_{YM} \right) \right],$$

where,

- \oint denotes “path integral integration,” and is only *formally* defined,
- $\omega \in \Omega^2(\mathcal{A}, \mathbb{R})$ is the Atiyah-Bott symplectic form on \mathcal{A} as defined above,
- $\Delta_{\mathcal{G}}$ and $\Delta_{\mathcal{A}}$ are *formally* the dimensions of \mathcal{G} and \mathcal{A} , and $\hbar \in \mathbb{R}^+$ a scalar parameter,
- $Z(\hbar)$ depends only on the volume of Σ , $\int_{\Sigma} \star 1$, where \star is defined by $\star 1 = \sigma$ for a symplectic form $\sigma \in \Omega^2(\Sigma)$. Fixing the volume and varying \hbar amounts to the same dependence.

At this point one observes that the Yang-Mills action (68) may be written as,

$$S_{YM}[A] = -\frac{1}{2}(\mu, \mu),$$

where $(\mu, \mu) := -\int_{\Sigma} \text{Tr}(\mu \wedge \star \mu)$, and is in the form of a moment map squared. This is not quite in the same form as the integrand for the *abelian* Duistermaat-Heckman theorem since in Eq. (63) the “action” is given by μ^Y and not (μ, μ) . This means that we can not a priori expect an analogous localization type result for the Yang-Mills path integral as in the Duistermaat-Heckman theorem. Observe that if we define,

$$\alpha(Y) := (-i)^{\Delta_{\mathcal{A}}/2} \exp(i\omega_{\text{Lie}\mathcal{G}}(Y)) = (-i)^{\Delta_{\mathcal{A}}/2} \exp(i\omega + i\langle \mu, Y \rangle),$$

then we may write,

$$(70) \quad Z(\hbar) = \frac{1}{\text{Vol}(\mathcal{G})} \int_{\text{Lie}\mathcal{G}} \left[\frac{dY}{2\pi} \right] \exp \left[-\frac{\hbar}{2}(Y, Y) \right] \oint_{\mathcal{A}} \alpha(Y),$$

$$(71) \quad = \frac{(-i)^{\Delta_{\mathcal{A}}/2}}{\text{Vol}(\mathcal{G})} \int_{\text{Lie}\mathcal{G}} \left[\frac{dY}{2\pi} \right] \oint_{\mathcal{A}} \exp \left[i\omega + i\langle \mu, Y \rangle - \frac{\hbar}{2}(Y, Y) \right].$$

The equivalence of Eq.’s (69) and (71) may be seen by doing the formal Gaussian integral over $\text{Lie}\mathcal{G}$ in (71). Let us formalize (70) and make the following,

Definition 15. Let $G \curvearrowright M$ be a smooth, oriented manifold with action of a Lie group G and let $\alpha \in \text{Map}^\infty(\mathfrak{g}, \Omega^\bullet(M))$. Define the equivariant integral of α as,

$$(72) \quad \oint_M^{\hbar} \alpha := \frac{1}{\text{Vol}(G)} \int_{\mathfrak{g}} [d\mathfrak{g}] \exp \left[-\frac{\hbar}{2}(Y, Y) \right] \int_M \alpha(Y),$$

where $[d\mathfrak{g}]$ is a measure on \mathfrak{g} and $\hbar \in \mathbb{R}^+$.

Note that we will formally take the standard measure on \mathfrak{g} and write $[d\mathfrak{g}] = \left[\frac{dY}{2\pi} \right]$, where we include a factor of 2π for each dimension of \mathfrak{g} . Our main point here is that there does exist a localization principal for quantities in the form of Eq. (70). The form of the expression for the localization will depend on a choice of form $\theta \in \Omega_G^\bullet(M)$. Given $\theta \in \Omega_G^\bullet(M)$, let,

$$\mathcal{C}^\theta := \{p \in M \mid \iota_{Y^\#} \theta|_p = 0, \forall Y \in \mathfrak{g}\}.$$

Write $\mathcal{C}^\theta = \bigcup_{\chi \in I} \mathcal{C}_\chi^\theta$, where \mathcal{C}_χ^θ are connected components of \mathcal{C}^θ enumerated by $\chi \in I$.

Proposition 16. [Wit92, §2] (*General non-abelian localization*) Let $G \curvearrowright M$ be a smooth, closed, oriented manifold with action of a Lie group G and let $\alpha \in \text{Map}^\infty(\mathfrak{g}, \Omega^\bullet(M))$ be an equivariantly closed form, $d_{\mathfrak{g}}\alpha = 0$, that has polynomial dependence on $Y \in \mathfrak{g}$ and $\theta \in \Omega_G^\bullet(M)$, then the equivariant integral of α can be written as a sum over contributions from the sets \mathcal{C}_χ^θ ,

$$\oint_M^\hbar \alpha = \sum_{\chi \in I} Z_\chi^\theta(\alpha, \hbar),$$

where $Z_\chi^\theta(\alpha, \hbar) \in \mathbb{C}$ depend only on \mathcal{C}_χ^θ and α , for each $\hbar \in \mathbb{R}^+$.

This is different from the Berline-Vergne theorem (14) for example, in that the localization depends on θ .

Proof. We follow [Wit92]. For all $t \in \mathbb{R}$, we may write,

$$\oint_M^\hbar \alpha = \oint_M^\hbar (\alpha \wedge e^{itd_{\mathfrak{g}}\theta}).$$

By definition,

$$\oint_M^\hbar \alpha \wedge e^{itd_{\mathfrak{g}}\theta} = \int_{\mathfrak{g}} \exp[-\frac{\hbar}{2}(Y, Y)][d\mathfrak{g}] \int_M \alpha(Y) \wedge e^{itd_{\mathfrak{g}}\theta(Y)}.$$

We consider $\int_M \alpha(Y) \wedge e^{itd_{\mathfrak{g}}\theta}$ and observe that,

$$e^{itd_{\mathfrak{g}}\theta(Y)} = 1 + itd_{\mathfrak{g}}\beta(Y),$$

for some $\beta \in \Omega_G^\bullet(M)$, by expanding the exponential. Since $\alpha(Y) \wedge d_{\mathfrak{g}}\beta(Y) = d_{\mathfrak{g}}(\alpha \wedge \beta)(Y)$ is equivariantly exact (since $d_{\mathfrak{g}}\alpha = 0$), we get,

$$\int_M \alpha(Y) \wedge e^{itd_{\mathfrak{g}}\theta(Y)} = \int_M \alpha(Y) + it \int_M d_{\mathfrak{g}}(\alpha \wedge \beta)(Y) = \int_M \alpha(Y),$$

by Stokes' theorem. At this point we make the simplifying assumption that $\theta \in \Omega^1(M)^G$, so that, as a map $\theta : \mathfrak{g} \rightarrow \Omega^1(M)$, θ is independent of $Y \in \mathfrak{g}$. Let $\{Y_q\}_{q=1}^d$ be an orthonormal basis for \mathfrak{g} and ϕ_1, \dots, ϕ_d coordinates so that $Y = \phi_1 Y_1 + \dots + \phi_d Y_d$ in general. Then,

$$(73) \quad \oint_M^\hbar \alpha = \int_{\mathfrak{g} \times M} [d\mathfrak{g}] \cdot \alpha(Y) \cdot \exp \left[itd\theta - it \sum_q \phi_q \theta(Y_q^\#) - \frac{\hbar}{2} \sum_q \phi_q^2 \right],$$

which follows from $d_{\mathfrak{g}}\theta(Y) \equiv d\theta - \iota_{Y^\#}\theta(Y) = d\theta - \sum_q \xi_q(Y)\theta(Y_q^\#)$, when $\theta \in \Omega^1(M)^G$. Next complete the square,

$$\frac{\hbar}{2}\phi_q^2 + it\theta(Y_q^\#)\phi_q = \frac{\hbar}{2} \left(\phi_q + \frac{it\theta(Y_q^\#)}{\hbar} \right)^2 + \frac{t^2\theta(Y_q^\#)^2}{2\hbar}.$$

Changing variables $\phi_q \mapsto \phi_q + \frac{it\theta(Y_q^\#)}{\hbar}$, and using translation invariance, we have,

$$(74) \quad \oint_M^\hbar \alpha = \int_{\mathfrak{g} \times M} [d\mathfrak{g}] \cdot \alpha(Y) \exp \left[itd\theta - \frac{\hbar}{2} \sum_q \phi_q^2 - \frac{t^2}{2\hbar} \sum_q \theta(Y_q^\#)^2 \right].$$

We take $[d\mathbf{g}]$ to be the Haar measure,

$$[d\mathbf{g}] = \left[\frac{dY}{2\pi} \right] = \frac{d\phi_1 d\phi_2 \cdots d\phi_d}{(2\pi)^d}.$$

We can perform the Gaussian integral,

$$\int_{\mathbb{R}} \frac{d\phi_q}{2\pi} \phi_q^{n_q} \exp \left[-\frac{\hbar}{2} \phi_q^2 \right] = \begin{cases} \sqrt{\frac{2\pi}{\hbar^{n_q+1}}} (1 \cdot 3 \cdot 5 \cdots (n_q + 1)), & n_q \in 2\mathbb{N}_0, \\ 0, & n_q \text{ odd.} \end{cases}$$

For simplicity, we assume that α is independent of Y . Then,

$$\oint_M^{\hbar} \alpha = \left(\frac{1}{2\pi\hbar} \right)^{d/2} \int_M \alpha \exp \left[itd\theta - \frac{t^2}{2\hbar} \sum_q \theta(Y_q^\sharp)^2 \right].$$

Clearly,

$$\mathcal{C}^\theta = \{p \in M \mid \iota_{Y_q^\sharp} \theta(p) = \theta(Y_q^\sharp)(p) = 0, 1 \leq q \leq d\}.$$

If $W \subseteq M$ is a compact subset of M with $\dim W = \dim M$, such that $W \cap \mathcal{C}^\theta = \emptyset$, then $\sum_q \theta(Y_q^\sharp)^2 > 0$ on W . This implies that,

$$\lim_{t \rightarrow \infty} \left(\frac{1}{2\pi\hbar} \right)^{d/2} \int_W \alpha \exp \left[itd\theta - \frac{t^2}{2\hbar} \sum_q \theta(Y_q^\sharp)^2 \right] = 0.$$

since the term of the form $\exp(-at^2)$ will dominate the convergence to 0 as $t \rightarrow \infty$ when integrated against $\exp(itd\theta)$. Let U_χ^θ be a G -equivariant tubular neighbourhood of \mathcal{C}_χ^θ and define,

$$Z_\chi^\theta(\alpha, \hbar) := \lim_{t \rightarrow \infty} \left(\frac{1}{2\pi\hbar} \right)^{d/2} \int_{U_\chi^\theta} \alpha \exp[itd\theta - \frac{t^2}{2\hbar} \sum_q \theta(Y_q^\sharp)^2].$$

Overall, we have shown that,

$$\oint_M^{\hbar} \alpha = \sum_{\chi \in I} Z_\chi^\theta(\alpha, \hbar).$$

Q.E.D.

Note that in the above proof we assumed for simplicity that α is independent of Y . The main result of the proposition is still true, however, if α has polynomial or exponential dependence on Y . In this case, the Y integral in (73) contributes at most an exponential t -dependence to the integral in (74), and this does not affect the localization result because this dependence is dominated by $\exp[-t^2]$ behavior in (74). Next, we have the following,

Proposition 17. *Let (M, ω, G, μ) be a smooth, closed Hamiltonian G -space and,*

$$\alpha(Y) := (-i)^{\Delta_M/2} \exp[i(\omega + \mu^Y)].$$

Then by judicious choice of $\theta \in \Omega^1(M)^G$, $\oint_M^{\hbar} \alpha$ localizes to the critical points of the function $f_\mu := \frac{1}{2}(\mu, \mu) : M \rightarrow \mathbb{R}$.

Proof. Recall that $\mathcal{C}^\theta = \{p \in M \mid \iota_{Y_q^\sharp} \theta(p) = 0, 1 \leq q \leq d\}$, where $\{Y_q\}_{q=1}^d$ is an orthonormal basis for \mathfrak{g} . We show for a good choice of θ that $\mathcal{C}^\theta = \text{Crit } f_\mu$. First, let $J : TM \rightarrow TM$ be an almost complex structure on M that is positive, i.e. so that the metric $g \in \text{Sym}^2 T^*M$, defined by,

$$g(u, v) := \omega(u, Jv),$$

is positive definite (such a J always exists and is in fact unique up to homotopy). Let $\tilde{J} : T^*M \rightarrow T^*M$ be the induced map $\tilde{J}(\tau)(v) := \tau(Jv)$ for $\tau \in T_p^*M$, $v \in T_pM$. We can identify \tilde{J} as the transpose of J and locally write,

$$\tilde{J}(a_i dx^i) = \sum_{i,k} J_{i,k} a_i dx^k.$$

Let $\tilde{g} : TM \rightarrow T^*M$ be the isomorphism defined by $\tilde{g}(v) := g(v, \cdot)$ and let $\tilde{\omega} : TM \rightarrow T^*M$ be defined by $\tilde{\omega}(v) := \omega(v, \cdot)$. Define,

$$(75) \quad \theta := \tilde{J} df_\mu.$$

We want to show that $\mathcal{C}^\theta := \text{Crit } f_\mu$. Clearly, $\text{Crit } f_\mu \subseteq \mathcal{C}^\theta$ since if $p \in \text{Crit } f_\mu$, then $df_\mu|_p = 0$, so $\mu|_p = 0$ and $\iota_{Y_q^\sharp} \theta(p) = 0$. To see that $\mathcal{C}^\theta \subseteq \text{Crit } f_\mu$, let $\mu_q := \mu^{Y_q} = (\mu, Y_q) \in C^\infty(M, \mathbb{R})$ and define $V := \sum_q \mu_q Y_q^\sharp \in \Gamma(TM)$. We may also express $\mu = \sum_q \mu_q \xi_q$, where ξ_q is the dual basis to Y_q , and also, $(\mu, \mu) = \sum_q \mu_q^2$. The moment map condition is $d\mu_q = \iota_{Y_q^\sharp} \omega = \tilde{\omega}(Y_q^\sharp)$, or $Y_q^\sharp = \tilde{\omega}^{-1} d\mu_q$, where $\tilde{\omega}^{-1} : T^*M \rightarrow TM$. So,

$$V = \sum_q \mu_q Y_q^\sharp = \sum_q \mu_q \tilde{\omega}^{-1} d\mu_q = \tilde{\omega}^{-1} \left[\sum_q \mu_q d\mu_q \right] = \tilde{\omega}^{-1} \left[\frac{1}{2} d(\mu, \mu) \right] = \tilde{\omega}^{-1}(df_\mu).$$

Now, by definition of θ and $\tilde{\omega}$,

$$(76) \quad \theta(V) = (\tilde{J} df_\mu)(\tilde{\omega}^{-1} df_\mu) = \omega^{-1}(df_\mu, \tilde{J} df_\mu).$$

Since $V = \sum_q \mu_q Y_q^\sharp$ and we assume that $\theta(Y_q^\sharp)(p) = 0$, then $\theta(V)(p) = 0$. Thus, $\omega^{-1}(df_\mu, \tilde{J} df_\mu)(p) = 0$, and $df_\mu(p) = 0$ by positivity of the associated metric g . Hence $p \in \mathcal{C}^\theta$ and $p \in \text{Crit } f_\mu$. Q.E.D.

Using θ in Prop. (17), an important critical component is $\mathcal{C}_0^\theta := \mu^{-1}(0)$. Then, by reduction to $\mathcal{M}_0 := \mu^{-1}(0)/\mathcal{G}$, the moduli space of flat connections on Σ , Witten [Wit92] shows for,

$$\alpha(Y) := (-i)^{\Delta_A/2} \exp[i(\omega + \mu^Y)],$$

that,

$$(77) \quad Z_0^\theta(\alpha, \hbar) = \int_{\mathcal{M}_0} \exp(\omega + \hbar \Theta),$$

where $\Theta \in H^4(\mathcal{M}_0, \mathbb{R})$. Note that there is a natural map $\text{pr}^* : H_\mathcal{G}^\bullet(\text{pt}) \simeq S(\text{Lie}(\mathcal{G})^*)^\mathcal{G} \rightarrow H_\mathcal{G}^\bullet(\mu^{-1}(0))$ induced by the projection $\text{pr} : \mu^{-1}(0) \rightarrow \text{pt}$, so that the class $-\langle Y, Y \rangle/2 \in H_\mathcal{G}^4(\text{pt})$ gives rise to the class $\Theta \in H^4(\mathcal{M}_0, \mathbb{R})$. A nice consequence is that one obtains $Z_0^\theta(\alpha, 0)$ as the symplectic volume of \mathcal{M}_0 . Also note that non-abelian localization in the finite dimensional setting is studied by Jeffrey-Kirwan [JK95]. In this paper, Jeffrey and Kirwan are able to reproduce Witten's result, Prop. (16), in [JK95, Theorem 1.1] and are also able to obtain a nice estimate [JK95, Theorem 1.2] for $\oint_M^\hbar \alpha$ as defined in Prop. (17) as a function of \hbar on the non-minimal critical point sets, \mathcal{C}_χ^θ for $\chi \neq 0$. [JK95] finds that $Z_\chi^\theta(\alpha, \hbar)$ is proportional

to a function that behaves like $\exp[-1/\hbar]$, times some function bounded by a polynomial in \hbar^{-1} , for $\chi \neq 0$.

5. SHIFT SYMMETRY AND THE ABELIAN PARTITION FUNCTION

Our goal in this section is to obtain a heuristic “shift invariant” expression for the abelian Chern-Simons partition function by decoupling one of the three components of the gauge field $A \in \mathcal{A}_P$ using a particular symmetry of our theory. The symmetry that we exhibit is directly associated to a choice of contact structure on our three manifold X and is called a “shift symmetry.” In this section, we will assume that (X, H) is a closed contact three manifold and $\kappa \in \Omega^1(X)$ is a contact one form, so that $\text{Ker}(\kappa) = H$. Note that every closed, orientable three manifold admits a contact structure [Mar71] and therefore the shift symmetry construction is a general symmetry that applies to any closed, orientable three-manifold. We note that the constructions in this section are largely heuristic and should be viewed as an *initial* step in obtaining a rigorous definition for a shift invariant expression of the abelian Chern-Simons partition function. For a finite dimensional analogue of the shift symmetry see Appendix (??). First we make the following,

Definition 18. *The action of the space of local shift symmetries \mathcal{S} on \mathcal{A}_P is defined by its variation δ_σ on a field $A \in \mathcal{A}_P$ by,*

$$\delta_\sigma A := \sigma \kappa,$$

where $\sigma \in \Omega^0(X, \mathfrak{t})$ is an arbitrary form and $\kappa \in \Omega^1(X)$ is a fixed contact form on X .

Note that since the shift symmetry is defined for arbitrary $\sigma \in \Omega^0(X, \mathfrak{t})$, it is independent of the choice of κ for the contact structure $H \subset TX$ since any two such contact forms must be related by a multiple of a non-vanishing scalar function on X . Clearly, the Chern-Simons action, $\text{CS}_{X,P}(A)$, does *not* respect the shift symmetry. That is,

$$(78) \quad \delta_\sigma \text{CS}_{X,P}(A) \neq 0,$$

for arbitrary $\sigma \in \Omega^0(X, \mathfrak{t})$. In order to study a shift invariant version of abelian Chern-Simons theory, we follow [BW05, §3.1] and introduce a new scalar field $\Phi \in \Omega^0(X, \mathfrak{t})$ such that,

$$\delta_\sigma \Phi = \sigma.$$

We postulate the scaling,

$$\Phi \rightarrow t^{-1}\Phi,$$

for a non-zero function $t \in C^\infty(X)$ whenever,

$$\kappa \rightarrow t\kappa,$$

so that $\kappa\Phi \in \Omega^1(X, \mathfrak{t})$ is invariant under the scaling by t and is a well defined form, independent of the choice of κ . Then for any principal \mathbb{T} -bundle P we define a new action,

$$\begin{aligned} \text{CS}_{X,P}(A, \Phi) &:= \text{CS}_{X,P}(A - \kappa\Phi) \\ &:= \int_X \alpha(\widehat{A - \kappa\Phi}), \\ (79) \quad &= \int_X \alpha(\hat{A} - \kappa\hat{\Phi}), \\ (80) \quad &= \text{CS}_{X,P}(A) - \frac{1}{4\pi} \int_X [2\kappa \wedge \text{Tr}(\hat{\Phi} \wedge F_{\hat{A}}) - \kappa \wedge d\kappa \text{Tr}(\hat{\Phi}^2)], \end{aligned}$$

where Eq. (79) follows from the definition of \hat{A} and $\hat{\Phi}$, where,

$$\hat{\Phi}|_{[p,h]} := \text{Ad}_{h^{-1}}(\iota_* \text{pr}_1^* \Phi|_h),$$

on $\hat{P} = P \times_{\mathbb{T}} SU(N+1)$. It is easy to see that the new action, $\text{CS}_{X,P}(A, \Phi)$, is invariant under the shift symmetry,

$$\begin{aligned} \delta_\sigma \text{CS}_{X,P}(A, \Phi) &= \frac{\delta \text{CS}_{X,P}}{\delta A}(A - \Phi\kappa) \cdot \delta_\sigma(A - \Phi\kappa), \\ &= \frac{\delta \text{CS}_{X,P}}{\delta A}(A - \Phi\kappa) \cdot (\delta_\sigma(A) - \delta_\sigma(\Phi)\kappa), \\ &= \frac{\delta \text{CS}_{X,P}}{\delta A}(A - \Phi\kappa) \cdot (\sigma\kappa - \sigma\kappa), \\ &= 0. \end{aligned}$$

Now define a “new” partition function,

$$(81) \quad \bar{Z}_{\mathbb{T}}(X, P, k) := \frac{1}{\text{Vol}(\mathcal{S})} \frac{1}{\text{Vol}(\mathcal{G})} \left(\frac{k}{4\pi^2} \right)^{\Delta_{\mathcal{G}}} \oint_{\mathcal{A}_P} \mathcal{D}A \mathcal{D}\Phi e^{ik \text{CS}_{X,P}(A, \Phi)},$$

where $\mathcal{D}\Phi$ is defined by the invariant, positive definite quadratic form, [BW05, Eq. 3.8],

$$(82) \quad (\Phi, \Phi) = - \int_X \langle \Phi, \Phi \rangle \kappa \wedge d\kappa.$$

As observed in [BW05], the new partition function of Eq. (81) should be identically equal to the original partition function defined for abelian Chern-Simons theory as in Eq. (20),

$$(83) \quad Z_{\mathbb{T}}(X, P, k) = \frac{1}{\text{Vol}(\mathcal{G})} \left(\frac{k}{4\pi^2} \right)^{\Delta_{\mathcal{G}}} \oint_{\mathcal{A}_P} \mathcal{D}A e^{ik \text{CS}_{X,P}(A)}.$$

This is seen by fixing $\Phi = 0$ using the shift symmetry, $\delta_\sigma \Phi = \sigma$, which will cancel the prefactor $\text{Vol}(\mathcal{S})$ from the resulting group integral over \mathcal{S} and yield exactly our original partition function,

$$Z_{\mathbb{T}}(X, P, k) = \frac{1}{\text{Vol}(\mathcal{G})} \left(\frac{k}{4\pi^2} \right)^{\Delta_{\mathcal{G}}} \oint_{\mathcal{A}_P} \mathcal{D}A e^{ik \text{CS}_{X,P}(A)}.$$

Thus, we obtain the heuristic result,

$$(84) \quad \bar{Z}_{\mathbb{T}}(X, k) = Z_{\mathbb{T}}(X, k).$$

On the other hand, we obtain another description of $\bar{Z}_{\mathbb{T}}(X, P, k)$ by integrating Φ out. We will briefly review this computation here. Our starting point is the formula for the shift reduced partition function in Eq. (80),

$$(85) \quad \bar{Z}_{\mathbb{T}}(X, P, k) = \frac{1}{\text{Vol}(\mathcal{S})} \frac{1}{\text{Vol}(\mathcal{G})} \left(\frac{k}{4\pi^2} \right)^{\Delta_{\mathcal{G}}} \oint_{\mathcal{A}_P} \mathcal{D}A \mathcal{D}\Phi e^{ik \text{CS}_{X,P}(A, \Phi)},$$

where,

$$(86) \quad \text{CS}_{X,P}(A, \Phi) = \text{CS}_{X,P}(A) - \frac{1}{4\pi} \int_X [2\kappa \wedge \text{Tr}(\hat{\Phi} \wedge F_{\hat{A}}) - \kappa \wedge d\kappa \text{Tr}(\hat{\Phi}^2)].$$

We formally complete the square with respect to $\hat{\Phi}$ as follows,

$$\begin{aligned}
& \int_X [\kappa \wedge d\kappa \operatorname{Tr}(\hat{\Phi}^2) - 2\kappa \wedge \operatorname{Tr}(\hat{\Phi} \wedge F_{\hat{A}})], \\
&= \int_X \left[\operatorname{Tr}(\hat{\Phi}^2) - \frac{2\kappa \wedge \operatorname{Tr}(\hat{\Phi} \wedge F_{\hat{A}})}{\kappa \wedge d\kappa} \right] \kappa \wedge d\kappa, \\
&= \int_X \operatorname{Tr} \left(\hat{\Phi}^2 - \frac{2\kappa \wedge F_{\hat{A}} \hat{\Phi}}{\kappa \wedge d\kappa} \right) \kappa \wedge d\kappa, \\
&= \int_X \operatorname{Tr} \left(\left[\hat{\Phi} - \frac{\kappa \wedge F_{\hat{A}}}{\kappa \wedge d\kappa} \right]^2 - \left[\frac{\kappa \wedge F_{\hat{A}}}{\kappa \wedge d\kappa} \right]^2 \right) \kappa \wedge d\kappa.
\end{aligned}$$

We then only need to compute the Gaussian integral,

$$\begin{aligned}
& \oint \mathcal{D}\Phi \exp \left[\frac{ik}{4\pi} \int_X \operatorname{Tr} \left(\left[\hat{\Phi} - \frac{\kappa \wedge F_{\hat{A}}}{\kappa \wedge d\kappa} \right]^2 \right) \kappa \wedge d\kappa \right], \\
&= \oint \mathcal{D}\Phi \exp \left[\frac{ik}{4\pi} \int_X \operatorname{Tr}(\hat{\Phi}^2) \kappa \wedge d\kappa \right], \\
&= \oint \mathcal{D}\Phi \exp \left[\frac{ik}{4\pi} \int_X \langle \Phi, \Phi \rangle \kappa \wedge d\kappa \right], \\
&= \oint \mathcal{D}\Phi \exp \left[-\frac{1}{2} (\Phi, T\Phi) \right],
\end{aligned}$$

where we take $T = \frac{ik}{2\pi} \mathbb{I}$ acting on the space of fields Φ and the inner product (Φ, Φ) is defined as in Eq. (82). We then formally get,

$$(87) \quad \oint \mathcal{D}\Phi \exp [-(\Phi, T\Phi)] = \sqrt{\frac{\pi \Delta \mathcal{G}}{\det T}},$$

$$(88) \quad = \left(-\frac{\pi i}{k} \right)^{\Delta \mathcal{G}/2},$$

where the quantity $\Delta \mathcal{G}$ is formally the dimension of the gauge group \mathcal{G} . Our new description of the partition function is now,

$$(89) \quad \bar{Z}_{\mathbb{T}}(X, P, k) = C \cdot \oint_{\mathcal{A}_P} \mathcal{D}A \exp \left[ik \left(\operatorname{CS}_{X,P}(A) - \frac{1}{4\pi} \int_X \frac{\operatorname{Tr}[(\kappa \wedge F_{\hat{A}})^2]}{\kappa \wedge d\kappa} \right) \right],$$

where $C = \frac{1}{\operatorname{Vol}(\mathcal{S})} \frac{1}{\operatorname{Vol}(\mathcal{G})} \left(\frac{-ik}{4\pi^2} \right)^{\Delta \mathcal{G}/2}$. We rewrite this partition function after identifying $\mathcal{A}_P = A_P + \Omega^1(X, \mathfrak{t})$ for a flat base point A_P in \mathcal{A}_P . We then obtain,

$$(90) \quad \bar{Z}_{\mathbb{T}}(X, P, k) = C_1 \cdot \oint_{\mathcal{A}_P} \mathcal{D}A \exp \left[\frac{ik}{4\pi} \left(\int_X \langle A \wedge dA \rangle - \int_X \frac{\langle (\kappa \wedge dA)^2 \rangle}{\kappa \wedge d\kappa} \right) \right],$$

where,

$$C_1 = \frac{e^{ik \operatorname{CS}_{X,P}(A_P)}}{\operatorname{Vol}(\mathcal{S}) \operatorname{Vol}(\mathcal{G})} \left(\frac{-ik}{4\pi^2} \right)^{\Delta \mathcal{G}/2}.$$

Note that the critical points of this action, up to the action of the shift symmetry, are precisely the flat connections, [BW05, Eq. 5.3]. We abuse notation and write $A \in T_{A_P} \mathcal{A}_P$. Let us define the notation,

$$(91) \quad 4\pi \overline{\text{CS}}(A) := \int_X \langle A \wedge dA \rangle - \int_X \frac{\langle (\kappa \wedge dA)^2 \rangle}{\kappa \wedge d\kappa},$$

for the new action that appears in the partition function. Also define,

$$(92) \quad \overline{\text{S}}(A) := \frac{1}{4\pi} \int_X \frac{\langle (\kappa \wedge dA)^2 \rangle}{\kappa \wedge d\kappa},$$

so that we may write,

$$(93) \quad \overline{\text{CS}}(A) = \text{CS}(A) - \overline{\text{S}}(A).$$

The primary virtue of Eq. (90) above is that it is heuristically equal to the original Chern-Simons partition function of Def. (5) and yet it is expressed in such a way that the action $\overline{\text{CS}}(A)$ is invariant under the shift symmetry. This means that $\overline{\text{CS}}(A + \sigma\kappa) = \overline{\text{CS}}(A)$ for all tangent vectors $A \in T_{A_P}(\mathcal{A}_P) \simeq \Omega^1(X, \mathfrak{t})$ and $\sigma \in \Omega^0(X, \mathfrak{t})$. We may naturally view $A \in \Omega^1(H, \mathfrak{t})$, the subset of $\Omega^1(X, \mathfrak{t})$ restricted to the contact distribution $H \subset TX$. If ξ denotes the Reeb vector field of κ , then $\Omega^1(H, \mathfrak{t}) = \{\alpha \in \Omega^1(X, \mathfrak{t}) \mid \iota_\xi \alpha = 0\}$. The remaining contributions to the partition function come from the orbits of \mathcal{S} in \mathcal{A}_P , which turn out to give a contributing factor of $\text{Vol}(\mathcal{S})$, [BW05, Eq. 3.32]. We thus reduce our integral to an integral over $\bar{\mathcal{A}}_P := \mathcal{A}_P / \mathcal{S}$ and obtain,

$$\begin{aligned} Z_{\mathbb{T}}(X, P, k) &= \frac{e^{ik \text{CS}_{X,P}(A_P)}}{\text{Vol}(\mathcal{G})} \int_{\bar{\mathcal{A}}_P} \bar{\mathcal{D}}A \exp \left[\frac{ik}{4\pi} \left(\int_X \langle A \wedge dA \rangle - \int_X \frac{\langle (\kappa \wedge dA)^2 \rangle}{\kappa \wedge d\kappa} \right) \right], \\ &= \frac{e^{ik \text{CS}_{X,P}(A_P)}}{\text{Vol}(\mathcal{G})} \int_{\bar{\mathcal{A}}_P} \bar{\mathcal{D}}A \exp [ik \overline{\text{CS}}(A)], \end{aligned}$$

where $\bar{\mathcal{D}}A$ denotes an appropriate quotient measure on $\bar{\mathcal{A}}_P$, and $A \in \Omega^1(H, \mathfrak{t}) \simeq T_{A_P} \bar{\mathcal{A}}_P$.

Remark 19. We now make a new heuristic definition of a partition function. Let $k \in \mathbb{Z}$ and (X, κ) a closed, oriented contact three-manifold. The shift reduced abelian Chern-Simons partition function, $\bar{Z}_{\mathbb{T}}(X, k)$, is the heuristic quantity,

$$(94) \quad \bar{Z}_{\mathbb{T}}(X, k) = \frac{1}{|W|} \sum_{[P] \in \text{Tors } H^2(X, \Lambda)} \bar{Z}_{\mathbb{T}}(X, P, k),$$

where $|W|$ is the order of the Weyl group for $\text{SU}(N+1)$ and,

$$(95) \quad \bar{Z}_{\mathbb{T}}(X, P, k) = \frac{e^{ik \text{CS}_{X,P}(A_P)}}{\text{Vol}(\mathcal{G})} \left(\frac{-ik}{4\pi^2} \right)^{\Delta \mathcal{G}/2} \int_{\bar{\mathcal{A}}_P} \bar{\mathcal{D}}A \exp [ik \overline{\text{CS}}(A)],$$

and $4\pi \overline{\text{CS}}(A) := \int_X \langle A \wedge dA \rangle - \int_X \frac{\langle (\kappa \wedge dA)^2 \rangle}{\kappa \wedge d\kappa}$ is the shift reduced Chern-Simons action.

6. MOMENT MAP SQUARED FORM OF THE PARTITION FUNCTION

Our starting point is a heuristically defined partition function. Let $k \in \mathbb{Z}$ and (X, κ) a closed, oriented contact three-manifold. Recall that the *shift reduced abelian Chern-Simons partition function*, $\bar{Z}_{\mathbb{T}}(X, k)$, defined in Remark (19) is the heuristic quantity,

$$(96) \quad \bar{Z}_{\mathbb{T}}(X, k) = \frac{1}{|W|} \sum_{[P] \in \text{Tors } H^2(X, \Lambda)} \bar{Z}_{\mathbb{T}}(X, P, k),$$

where $|W|$ is the order of the Weyl group for $\text{SU}(N+1)$ and,

$$(97) \quad \bar{Z}_{\mathbb{T}}(X, P, k) = \frac{e^{ik \text{CS}_{X,P}(A_P)}}{\text{Vol}(\mathcal{G})} \left(\frac{-ik}{4\pi^2} \right)^{\Delta \mathcal{G}/2} \oint_{\bar{\mathcal{A}}_P} \bar{\mathcal{D}}A \exp [ik \overline{\text{CS}}(A)],$$

and $4\pi \overline{\text{CS}}(A) := \int_X \langle A \wedge dA \rangle - \int_X \frac{\langle \kappa \wedge dA \rangle^2}{\kappa \wedge d\kappa}$ is the *shift reduced Chern-Simons action*. Our next objective is to determine a moment map μ for a group action such that the shift reduced action may be identified as the moment map squared, $\overline{\text{CS}}(A) = (\mu, \mu)$. First we observe that the gauge group \mathcal{G} itself cannot act in a Hamiltonian fashion such that $\overline{\text{CS}}(A) = (\mu, \mu)$ since $\overline{\text{CS}}(A)$ is not invariant under “large” gauge transformations. We therefore restrict to the connected component of the gauge group containing the identity element, \mathcal{G}_0 . Let $Y \in \text{Lie}(\mathcal{G}_0) \simeq \Omega^0(X, \mathfrak{t})$, and let $Y^\# := d^H Y \in \Gamma(T\bar{\mathcal{A}})$ denote the vector field generated by Y on $\bar{\mathcal{A}}$. By definition of the symplectic form Ω on $\bar{\mathcal{A}}$, we have,

$$(\iota_{Y^\#} \Omega)(\delta A) = - \int_X \kappa \wedge \langle dY \wedge \delta A \rangle.$$

If,

$$\mu^Y(A) := \int_X \kappa \wedge \langle Y \wedge F_A \rangle - \int_X d\kappa \wedge \langle Y \wedge A \rangle,$$

then one can show that the moment map equation is satisfied,

$$d\mu^Y = \iota_{Y^\#} \Omega,$$

using integration by parts. We observe that the \mathcal{G}_0 action on $\bar{\mathcal{A}}$ is not Hamiltonian, however. This may be checked by computing the Poisson bracket,

$$\begin{aligned} \{\mu^{Y_1}, \mu^{Y_2}\} &= \Omega(d^H Y_1, d^H Y_2), \\ &= - \int_X \kappa \wedge \langle dY_1 \wedge dY_2 \rangle, \\ &= \int_X \kappa \wedge \langle [Y_1, Y_2] \wedge F_A \rangle - \int_X d\kappa \wedge \langle Y_1 \wedge dY_2 \rangle, \\ &= \mu^{[Y_1, Y_2]} - \int_X d\kappa \wedge \langle Y_1 \wedge dY_2 \rangle. \end{aligned}$$

The obstruction for the map μ to determine a moment map is given by the cocycle,

$$(98) \quad \begin{aligned} c(Y_1, Y_2) &:= \{\mu^{Y_1}, \mu^{Y_2}\} - \mu^{[Y_1, Y_2]}, \\ &= - \int_X d\kappa \wedge \langle Y_1 \wedge dY_2 \rangle, \\ &= - \int_X \langle Y_1 \wedge \mathcal{L}_\xi Y_2 \rangle \kappa \wedge d\kappa, \end{aligned}$$

which clearly does not vanish in general. Following the ideas of [BW05, §3.4], one may obtain a Hamiltonian action by considering a central extension $\tilde{\mathcal{G}}_0$ by $\mathbb{U}(1)$ of the group \mathcal{G}_0 determined by the cocycle $c(Y_1, Y_2)$,

$$\mathbb{U}(1) \rightarrow \tilde{\mathcal{G}}_0 \rightarrow \mathcal{G}_0.$$

As in [BW05], we assume that the central $\mathbb{U}(1)$ subgroup of $\tilde{\mathcal{G}}_0$ acts trivially on $\bar{\mathcal{A}}$ and the moment map for the central generator $(0, a)$ of the Lie algebra is constant. We then see that the new moment map for the action of $\tilde{\mathcal{G}}_0$ on $\bar{\mathcal{A}}$,

$$\mu^{(Y,a)}(A) := \int_X \kappa \wedge \langle Y \wedge F_A \rangle - \int_X d\kappa \langle Y \wedge A \rangle + a,$$

is Hamiltonian,

$$\{\mu^{(Y_1, a_1)}, \mu^{(Y_2, a_2)}\} = \mu^{[(Y_1, a_1), (Y_2, a_2)]},$$

where,

$$[(Y_1, a_1), (Y_2, a_2)] := ([Y_1, Y_2], c(Y_1, Y_2)).$$

In order to cast the action $\overline{\text{CS}}$ into a moment map squared form the Lie algebra of the Hamiltonian group must admit a non-degenerate invariant inner product. The group $\tilde{\mathcal{G}}_0$ does *not* admit such an inner product, however. Following [BW05], this problem may be solved by considering the Seifert action of $\mathbb{U}(1)$ on X . Since the action of $\mathbb{U}(1)$ on X induces an action on $\tilde{\mathcal{G}}_0$, we naturally consider the semidirect product $\mathbb{U}(1) \ltimes \tilde{\mathcal{G}}_0$, which admits the non-degenerate invariant inner product on the Lie algebra of $\mathbb{U}(1) \ltimes \tilde{\mathcal{G}}_0$,

$$(99) \quad ((p_1, Y_1, a_1), (p_2, Y_2, a_2)) = \left(- \int_X \langle Y_1 \wedge Y_2 \rangle \kappa \wedge d\kappa \right) - p_1 a_2 - p_2 a_1.$$

The Lie bracket on $\text{Lie}(\mathbb{U}(1) \ltimes \tilde{\mathcal{G}}_0)$ is also given by,

$$(100) \quad [(p_1, Y_1, a_1), (p_2, Y_2, a_2)] := (0, [Y_1, Y_2] + p_1 \mathcal{L}_\xi Y_2 - p_2 \mathcal{L}_\xi Y_1, c(Y_1, Y_2)).$$

One can show that the vector field on \mathcal{A} generated by an element $\mathbf{Y} = (p, Y, a) \in \text{Lie}(\mathbb{U}(1) \ltimes \tilde{\mathcal{G}}_0)$ is given by,

$$(101) \quad \mathbf{Y}^\#(A) = dY + p \mathcal{L}_\xi A.$$

Using (101), one can see that the moment map in direction of the generator $\mathbf{Y}_p := (p, 0, 0)$ may be given by,

$$\mu^{\mathbf{Y}_p}(A) = -\frac{1}{2}p \int_X \kappa \wedge \langle \mathcal{L}_\xi A \wedge A \rangle.$$

Clearly, $\mu^{\mathbf{Y}_p}(A)$ is invariant under the shift symmetry and descends to $\bar{\mathcal{A}}$. We claim that the action of $\mathbb{U}(1) \ltimes \tilde{\mathcal{G}}_0$ on $\bar{\mathcal{A}}$ defined above is Hamiltonian with moment map,

$$(102) \quad \mu^{\mathbf{Y}}(A) = -\frac{1}{2}p \int_X \kappa \wedge \langle \mathcal{L}_\xi A \wedge A \rangle + \int_X \kappa \wedge \langle Y \wedge dA \rangle - \int_X d\kappa \wedge \langle Y \wedge A \rangle + a.$$

Let $\mathbf{Y}_Y := (0, Y, 0) \in \text{Lie}(\mathbb{U}(1) \ltimes \tilde{\mathcal{G}}_0)$. In order to show that (102) is a moment map we need only compute,

$$\{\mu^{\mathbf{Y}_p}, \mu^{\mathbf{Y}_Y}\},$$

since this is the only non-trivial Poisson bracket that remains to be checked. We compute,

$$\begin{aligned}
\{\mu^{\mathbf{Y}_p}, \mu^{\mathbf{Y}_Y}\} &= \Omega(p\mathcal{L}_\xi A, dY), \\
&= -p \int_X \kappa \wedge \langle \mathcal{L}_\xi A \wedge dY \rangle, \\
&= p \int_X \kappa \wedge \langle \mathcal{L}_\xi Y \wedge dA \rangle - p \int_X d\kappa \wedge \langle \mathcal{L}_\xi Y \wedge A \rangle, \\
&= \mu^{\mathbf{Y}_{p\mathcal{L}_\xi Y}},
\end{aligned}$$

where $\mathbf{Y}_{p\mathcal{L}_\xi Y} := (0, p\mathcal{L}_\xi Y, 0)$. By definition of the Lie bracket in (100), our last computation shows that the moment map condition is satisfied. We therefore take $\mathcal{H} := \mathbb{U}(1) \ltimes \tilde{\mathcal{G}}_0$ to be the Hamiltonian group for abelian Chern-Simons theory.

We now claim that the action $\overline{\text{CS}}$ may be expressed in moment map squared form (μ, μ) for the moment map defined in (102). Let $\mathfrak{H} := \text{Lie}(\mathcal{H})$ and $\langle \langle \cdot, \cdot \rangle \rangle : \mathfrak{H}^* \otimes \mathfrak{H} \rightarrow \mathbb{R}$ denote the dual pairing on \mathfrak{H} . Note that we have implicitly been using the notation $\mu^{\mathbf{Y}}$ to mean,

$$\mu^{\mathbf{Y}} = \langle \langle \mu, \mathbf{Y} \rangle \rangle,$$

where we view $\mu \in \mathfrak{H}^*$. We make the identification $\mathfrak{H}^* \simeq \mathfrak{H}$ via the pairing (\cdot, \cdot) defined in (99). Let $\hat{\mu} \in \mathfrak{H}$ be defined by $\mu = (\hat{\mu}, \cdot)$. We abuse notation and write,

$$(\mu, \mu) := (\hat{\mu}, \hat{\mu}).$$

One can check that,

$$(103) \quad \hat{\mu} = \left(-1, - \left(\frac{\kappa \wedge dA - d\kappa \wedge A}{\kappa \wedge d\kappa} \right), \frac{1}{2} \int_X \kappa \wedge \langle \mathcal{L}_\xi A \wedge A \rangle \right),$$

and indeed,

$$(104) \quad \mu^{\mathbf{Y}} = \langle \langle \mu, \mathbf{Y} \rangle \rangle = (\hat{\mu}, \mathbf{Y}),$$

for all $\mathbf{Y} \in \mathfrak{H}$. Thus, by definition,

$$\begin{aligned}
(\mu, \mu) &= (\hat{\mu}, \hat{\mu}) = \langle \langle \mu, \hat{\mu} \rangle \rangle = \mu^{\hat{\mu}}, \\
(105) \quad &= \int_X \kappa \wedge \langle \mathcal{L}_\xi A \wedge A \rangle - \int_X \kappa \wedge d\kappa \left\langle \left(\frac{\kappa \wedge dA - d\kappa \wedge A}{\kappa \wedge d\kappa} \right)^2 \right\rangle.
\end{aligned}$$

Now consider the term in (105) given by, $\int_X \kappa \wedge \langle \mathcal{L}_\xi A \wedge A \rangle$. Using Cartan's formula for the Lie derivative, $\mathcal{L}_\xi = \{\iota_\xi, d\}$, we may write,

$$\int_X \kappa \wedge \langle \mathcal{L}_\xi A \wedge A \rangle = \int_X \kappa \wedge \langle \{\iota_\xi, d\} A \wedge A \rangle.$$

Using the fact that,

$$\iota_\xi A = \frac{A \wedge d\kappa}{\kappa \wedge d\kappa},$$

and using integration by parts we obtain,

$$\begin{aligned}
 \int_X \kappa \wedge \langle \mathcal{L}_\xi A \wedge A \rangle &= \int_X [\langle A \wedge dA \rangle - \kappa \wedge \langle dA \wedge \iota_\xi A \rangle + d\kappa \wedge \langle \iota_\xi A \wedge A \rangle] \\
 &\quad - \int_X [\kappa \wedge \langle \iota_\xi A \wedge dA \rangle], \\
 (106) \qquad &= \int_X \left[\langle A \wedge dA \rangle - 2\kappa \wedge \left\langle \left(\frac{d\kappa \wedge A}{\kappa \wedge d\kappa} \wedge dA \right) \right\rangle \right] + \\
 &\quad + \int_X \left[d\kappa \wedge \left\langle \left(\frac{d\kappa \wedge A}{\kappa \wedge d\kappa} \wedge A \right) \right\rangle \right].
 \end{aligned}$$

After expanding the term,

$$\int_X \kappa \wedge d\kappa \left\langle \left(\frac{\kappa \wedge dA - d\kappa \wedge A}{\kappa \wedge d\kappa} \right)^2 \right\rangle,$$

in (105), one finds that the correct terms cancel in (106) to yield,

$$\begin{aligned}
 (\mu, \mu) &= \int_X \langle A \wedge dA \rangle - \int_X \frac{1}{\kappa \wedge d\kappa} \langle (\kappa \wedge dA)^2 \rangle, \\
 &= 4\pi \overline{\text{CS}}(A),
 \end{aligned}$$

as desired. As in [BW05, §3.3], we observe that the path integral measure $\bar{\mathcal{D}}A$ in (97) should be identified with the corresponding symplectic measure $\exp(\Omega)$ in the path integral and we may write,

$$(107) \qquad \bar{Z}_{\mathbb{T}}(X, P, k) = \frac{e^{ik \text{CS}_{X,P}(A_P)}}{\text{Vol}(\mathcal{G})} \left(\frac{-ik}{4\pi^2} \right)^{\Delta \mathcal{G}/2} \oint_{\bar{\mathcal{A}}_P} \exp \left[\Omega + \frac{ik}{4\pi} (\mu, \mu) \right].$$

7. NON-ABELIAN LOCALIZATION FOR ABELIAN CHERN-SIMONS THEORY

Our starting point in this section is the main result of §6, which expresses the abelian partition function in *moment map squared* form,

$$(108) \qquad \bar{Z}_{\mathbb{T}}(X, P, k) = \frac{e^{ik \text{CS}_{X,P}(A_P)}}{\text{Vol}(\mathcal{G})} \left(\frac{-ik}{4\pi^2} \right)^{\Delta \mathcal{G}/2} \oint_{\bar{\mathcal{A}}_P} \exp \left[\Omega + \frac{ik}{4\pi} (\mu, \mu) \right].$$

Starting with this description of the partition function we follow the main arguments of [BW05], which must be adapted slightly for an abelian structure group, to arrive at the final rigorous definition 23.

7.1. A Two-Dimensional Description of the Abelian Partition Function. As observed in Prop. 17, $\bar{Z}_{\mathbb{T}}(X, P, k)$ localizes to the critical points of the function,

$$\begin{aligned}
 f_\mu := (\mu, \mu) &= \int_X \langle A \wedge dA \rangle - \int_X \frac{1}{\kappa \wedge d\kappa} \langle (\kappa \wedge dA)^2 \rangle, \\
 &= 4\pi \overline{\text{CS}}(A).
 \end{aligned}$$

Observe that the critical points of f_μ satisfy the equation of motion,

$$(109) \qquad dA - (\star_H dA) \wedge d\kappa - \kappa \wedge d \star_H dA = 0.$$

Recall the following,

Definition 20. Let $(X, \phi, \xi, \kappa, g)$ be a contact metric three manifold and define the horizontal Hodge star to be the operator,

$$\star_H : \Omega^q(X, \mathfrak{t}) \rightarrow \Omega^{2-q}(H, \mathfrak{t}) \quad q = 0, 1, 2,$$

defined for $\beta \in \Omega^q(X, \mathfrak{t})$ by,

$$(110) \quad \star_H \beta = \star(\kappa \wedge \beta),$$

where \star is the usual Hodge star operator on forms for the metric $g = \kappa \otimes \kappa + d\kappa(\cdot, \phi \cdot)$ on X .

Since f_μ is invariant under the shift symmetry, it is clear that (109) is also invariant under this symmetry. Thus, the critical points of f_μ can be classified as solutions of (109) relative to any convenient gauge choice for the shift symmetry. Observe that the quantity $\kappa \wedge dA$ transforms as,

$$\kappa \wedge dA \rightarrow \kappa \wedge dA + \sigma \kappa \wedge d\kappa,$$

under the shift symmetry for arbitrary $\sigma \in \Omega^1(X, \mathfrak{t})$. Thus, a valid gauge condition is given by setting,

$$(111) \quad \star_H dA = 0,$$

since given arbitrary $A \in \Omega^1(X, \mathfrak{t})$, we may set $\sigma := \star_H dA$ uniquely so that $A - \sigma \kappa$ satisfies (111). In this gauge the solutions of the equation of motion (109) are precisely the flat connections.

Remark 21. Note that the normalization $\bar{Z}_\mathbb{T}(X, P, k)$ in (108) needs to be revised slightly to take into account the fact that we have replaced the gauge group \mathcal{G} with the group $\mathcal{H} := \mathbb{U}(1) \ltimes \tilde{\mathcal{G}}_0$ in our considerations. In fact, we should formally replace \mathcal{G} with $\mathcal{H}' := \mathbb{U}(1) \ltimes \tilde{\mathcal{G}}$, where $\tilde{\mathcal{G}}$ represents a central extension of the full gauge group by $\mathbb{U}(1)$. We therefore wish to formally consider,

$$(112) \quad \bar{Z}'_\mathbb{T}(X, P, k) := \frac{e^{ik \text{CS}_{X,P}(A_P)}}{\text{Vol}(\mathcal{H}')} \left(\frac{-ik}{4\pi^2} \right)^{\Delta_{\mathcal{H}'}/2} \oint_{\bar{\mathcal{A}}_P} \exp \left[\Omega + \frac{ik}{4\pi}(\mu, \mu) \right],$$

where $\Delta_{\mathcal{H}'} = \dim \mathcal{H}'$. As observed in [BW05, Eq. 5.10] this results in a difference between $\bar{Z}_\mathbb{T}(X, P, k)$ and $\bar{Z}'_\mathbb{T}(X, P, k)$ by the finite multiplicative factor,

$$(113) \quad \frac{\text{Vol}(\mathcal{H}')}{i \text{Vol}(\mathcal{G})} \left(\frac{-ik}{4\pi^2} \right)^{\frac{1}{2}(\Delta_{\mathcal{G}} - \Delta_{\mathcal{H}'})} = \text{Vol}(U(1)^2) \cdot \frac{4\pi^2}{k}.$$

The technique of non-abelian localization applies more directly to $\bar{Z}'_\mathbb{T}(X, P, k)$ and this is the quantity that we will consider. In the end we must multiply our results by the factor (113) to recover the Chern-Simons path integral $\bar{Z}_\mathbb{T}(X, P, k)$.

Next, we focus on giving a “two-dimensional” description of the local symplectic geometry in $\bar{\mathcal{A}}_P$ around a critical point of the Chern-Simons action analogous to [BW05, §5.1]. For this it will be convenient to choose a particular gauge for the shift symmetry corresponding to the gauge condition,

$$(114) \quad \iota_\xi A = 0.$$

Clearly (114) defines a good gauge condition since if $\iota_\xi A = 0$, then $\iota_\xi(A + \sigma \kappa) = \sigma \neq 0$, i.e. the condition (114) picks out a unique representative for each orbit of the shift symmetry. Note that this gauge condition is defined on the tangent space $T_{A_P} \bar{\mathcal{A}}_P$ about a flat connection

A_P . Of course, this is the natural gauge that we will implicitly work with throughout, and gives the identification,

$$(115) \quad T_{A_P} \bar{\mathcal{A}}_P \simeq \Omega^1(H, \mathfrak{t}),$$

where $\Omega^1(H, \mathfrak{t}) := \{A \in \Omega^1(X, \mathfrak{t}) \mid \iota_\xi A = 0\}$.

Remark 22. *In order to proceed it is natural to make a simplifying assumption regarding the contact structure that we have been working with. Indeed, we have made no assumption about the contact structure up until this point and all of our considerations have been completely valid for the general case. We now make the assumption that the contact structure is such that the Reeb vector field generates a locally free $\mathbb{U}(1)$ action. We make this assumption so that we may Fourier decompose the tangent space $T_{A_P} \bar{\mathcal{A}}_P \simeq \Omega^1(H, \mathfrak{t})$ with respect to the natural $\mathbb{U}(1)$ action. It turns out that the closed three manifolds that admit a locally free $\mathbb{U}(1)$ action are precisely the (quasi-regular) Sasakian manifolds [BG08, Theorem 7.5.2]. We note that the existence of a Sasakian metric greatly simplifies calculations.*

Thus, assume that $(X, \kappa, \xi, \phi, g)$ defines a quasi-regular Sasakian structure. See [BG08] for more details on Sasakian manifolds. Given that the Reeb vector field ξ generates a locally free $\mathbb{U}(1)$ action, we naturally decompose the tangent space $T_{A_P} \bar{\mathcal{A}}_P \simeq \Omega^1(H, \mathfrak{t})$ with respect to this action and write,

$$(116) \quad A = \sum_{l \in \mathbb{Z}} A_l,$$

where $A_l \in \Omega^1(H, \mathfrak{t})$ are eigenmodes of the Lie derivative \mathcal{L}_ξ ,

$$(117) \quad \mathcal{L}_\xi A_l = -2\pi i l A_l.$$

We also decompose $Y \in \text{Lie}(\mathcal{G}) \simeq \Omega^0(X, \mathfrak{t})$ with respect to the $\mathbb{U}(1)$ action,

$$(118) \quad Y = \sum_{l \in \mathbb{Z}} Y_l,$$

where,

$$(119) \quad \mathcal{L}_\xi Y_l = -2\pi i l Y_l.$$

Let,

$$(120) \quad \mathcal{L} := X \times_{\mathbb{U}(1)} \mathbb{C},$$

denote the complex line V-bundle over Σ associated to the standard representation on \mathbb{C} . We view the eigenmodes $A_l \in \Omega^1(H, \mathfrak{t})$ as naturally corresponding to elements of,

$$\Omega^1(\Sigma, \mathcal{L}^l \otimes \mathfrak{t}) := \Gamma(\Sigma, T^*\Sigma \otimes \mathcal{L}^l \otimes \mathfrak{t}),$$

and formally decompose the tangent space $T_{A_P} \bar{\mathcal{A}}_P$ at A_P as,

$$(121) \quad T_{A_P} \bar{\mathcal{A}}_P = \bigoplus_{l \in \mathbb{Z}} \Omega^1(\Sigma, \mathcal{L}^l \otimes \mathfrak{t}).$$

Similarly, we decompose $\text{Lie}(\mathcal{G})$,

$$(122) \quad \text{Lie}(\mathcal{G}) = \bigoplus_{l \in \mathbb{Z}} \Omega^0(\Sigma, \mathcal{L}^l \otimes \mathfrak{t}).$$

7.2. Non-Abelian Localization Applied in Abelian Chern-Simons Theory. As in [BW05, §4.2], a local symplectic neighborhood of $\bar{\mathcal{A}}_P$ near \mathcal{M}_P , say N , is an equivariant fibration,

$$(123) \quad F \rightarrow N \rightarrow \mathcal{M}_P,$$

where the fibre F takes the form,

$$(124) \quad F = \mathcal{H} \times_{\mathcal{H}_0} (\mathfrak{h} \ominus \mathfrak{h}_0 \ominus \mathcal{E}_0 \oplus \mathcal{E}_1),$$

and \mathcal{H}_0 , \mathcal{E}_0 , \mathcal{E}_1 remain to be identified. Note that the symbol “ \ominus ” is to be interpreted in the sense of K-theory. In (124), we have $\mathcal{H} = \mathbb{U}(1) \ltimes \tilde{\mathcal{G}}_0$ as before. As in [BW05, Eq. 5.27], \mathcal{H}_0 is the subgroup of \mathcal{H} that fixes A_P and in general is of the form,

$$(125) \quad \mathcal{H}_0 = \mathbb{U}(1) \times \mathbb{U}(1) \times I,$$

where $\mathbb{U}(1)$ arises as the central extension group for $\tilde{\mathcal{G}}_0$, $\mathbb{U}(1)$ arises as the group acting on \mathcal{A}_P induced from the geometric action on X , and $I \simeq \mathbb{T}$ is the isotropy subgroup of A_P . As in [BW05, Eq. 5.29] we may identify \mathcal{E}_0 and \mathcal{E}_1 as,

$$(126) \quad \mathcal{E}_0 = \bigoplus_{l \in \mathbb{N}} H_{\bar{\partial}}^0(\Sigma, (\mathcal{L}^l \oplus \mathcal{L}^{-l}) \otimes \mathfrak{t}),$$

and,

$$(127) \quad \mathcal{E}_1 = \bigoplus_{l \in \mathbb{N}} H_{\bar{\partial}}^1(\Sigma, (\mathcal{L}^l \oplus \mathcal{L}^{-l}) \otimes \mathfrak{t}).$$

Now we express the partition function in (108),

$$(128) \quad \bar{Z}_{\mathbb{T}}(X, P, k) = \frac{e^{ik \text{CS}_{X,P}(A_P)}}{\text{Vol}(\mathcal{G})} \left(\frac{-ik}{4\pi^2} \right)^{\Delta \mathcal{G}/2} \oint_{\bar{\mathcal{A}}_P} \exp \left[\Omega + \frac{ik}{4\pi}(\mu, \mu) \right],$$

in a form that non-abelian localization can be more directly applied as in Prop. 16. We write the following for $\bar{Z}_{\mathbb{T}}(X, P, k)$,

$$(129) \quad \frac{e^{ik \text{CS}_{X,P}(A_P)}}{\text{Vol}(\mathcal{G})} \int_{\text{Lie } \mathcal{G}} \left[\frac{dY}{2\pi} \right] \oint_{\mathcal{A}} \exp \left[\Omega + i\langle \mu, Y \rangle - \frac{4\pi i}{k}(Y, Y) \right],$$

where the equivalence of Eq.’s (128) and (129) may be seen by doing the formal Gaussian integral over $\text{Lie } \mathcal{G}$ in (129). As in the proof of Prop. (17) we begin the localization computation by choosing θ as in (75),

$$(130) \quad \theta := \tilde{J}df_{\mu},$$

where $f_{\mu} := \frac{1}{2}(\mu, \mu)$. We then write the integral in (129) over N and this gives the following for $\bar{Z}_{\mathbb{T}}(X, P, k)$ in (123) as,

$$(131) \quad \frac{4\pi^2 \text{Vol}(U(1)^2)}{k \text{Vol}(\mathcal{H})} e^{ik \text{CS}_{X,P}(A_P)} \oint_{\mathfrak{H} \times N} \left[\frac{dY}{2\pi} \right] \exp \left[\Omega + i\langle \mu, Y \rangle - \frac{4\pi i}{k}(Y, Y) + t \cdot D\theta \right],$$

where $\mathfrak{H} = \text{Lie } \mathcal{H}$, $D := d_{\text{Lie } \mathcal{G}}$ denotes the formal equivariant derivative as in (65), $t \in \mathbb{R}$ and we include the normalization factor $\text{Vol}(U(1)^2)$ as in (113). The goal is now to reduce the integral over $\mathfrak{H} \times N$ in (131) to an integral over \mathcal{M}_P . This reduction problem is discussed briefly for Yang-Mills theory at the end of §4. We start by observing that the fibre F in (123) may be modelled on the cotangent bundle $T^*\mathcal{H}$, so that N equivariantly retracts onto a principal \mathcal{H} -bundle $P_{\mathcal{H}}$ over the moduli space \mathcal{M}_P . Following the argument of [Wit92] we

observe that if \mathcal{H} acts freely on $P_{\mathcal{H}}$ that the equivariant cohomology of the total space $P_{\mathcal{H}}$ may be identified with the ordinary cohomology of the quotient $P_{\mathcal{H}}/\mathcal{H} \simeq \mathcal{M}_P$. This allows us to identify the equivariant forms $\Omega + i\langle\mu, Y\rangle$ and (Y, Y) with the pullback of ordinary forms on \mathcal{M}_P .

In Chern-Simons theory, \mathcal{H} does not act freely on N , however. We may still follow the same reasoning as in Yang-Mills theory [Wit92] by taking into account that the subgroup \mathcal{H}_0 in (125) acts on N with fixed points. We then obtain an equivariant retraction N_0 of N as a bundle with fibre $\mathcal{H}/\mathcal{H}_0$,

$$(132) \quad \mathcal{H}/\mathcal{H}_0 \rightarrow N_0 \rightarrow \mathcal{M}_P.$$

Following the same argument of [BW05, Eq. 5.109], we may identify the \mathcal{H} equivariant forms,

$$\Omega + i\langle\mu, Y\rangle, \quad (Y, Y),$$

on N with the corresponding \mathcal{H}_0 equivariant forms,

$$\Omega + ia, \quad n\Theta + pa,$$

on \mathcal{M}_P , respectively, via pullback by a map,

$$(133) \quad \text{pr} : N \rightarrow \mathcal{M}_P.$$

Note that $p, a \in \mathfrak{H}_0$, $\Theta \in H^4(\mathcal{M}_P)$, $n = c_1(X)$ is the first Chern number of X as a bundle over Σ , and we have abused notation by writing Ω to represent the corresponding forms on both N and \mathcal{M}_P . Note that for simplicity we assume that $\mathbb{U}(1)$ acts freely on X . Let,

$$\tilde{K} := \frac{4\pi^2}{k} \frac{\text{Vol}(U(1)^2)}{\text{Vol}(\mathcal{H})} e^{ik \text{CS}_{X,P}(A_P)}.$$

We may then write,

$$(134) \quad \bar{Z}_{\mathbb{T}}(X, P, k) = \tilde{K} \oint_{\mathfrak{H} \times N} \left[\frac{dY}{2\pi} \right] \exp \left[\text{pr}^* \Omega + ia \left(1 - \frac{2\pi}{k} p \right) + \frac{2\pi i n}{k} \text{pr}^* \Theta + t \cdot D\theta \right],$$

exactly as in [BW05, Eq. 5.111]. Next, we decompose \mathfrak{H} as,

$$\mathfrak{H} = (\mathfrak{H} \ominus \mathfrak{H}_0) \oplus \mathfrak{H}_0,$$

and integrate over the variables a, p spanning \mathfrak{H}_0 in (134). Setting $\epsilon := \frac{2\pi}{k}$, the integral over a will produce a delta function that sets $p = \frac{1}{\epsilon}$, and we obtain the following for $\bar{Z}_{\mathbb{T}}(X, P, k)$,

$$(135) \quad \frac{\text{Vol}(U(1)^2)}{\text{Vol}(\mathcal{H})} e^{ik \text{CS}_{X,P}(A_P)} \oint_{(\mathfrak{H} \ominus \mathfrak{H}_0) \times N} \left[\frac{dY}{2\pi} \right] \exp \left[\text{pr}^* \Omega + i\epsilon n \text{pr}^* \Theta + t \cdot D\theta|_{\{p=\frac{1}{\epsilon}\}} \right].$$

Note that we will drop the term $\frac{1}{k}$ that occurs in the definition of \tilde{K} above and implicitly redefine the partition function to take this into account. As in [Wit92], the only term which is not pull back from \mathcal{M}_P is the localization term $t \cdot D\theta$, and we are left to perform the computation of the $t \cdot D\theta$ dependent part of (135) over F . In Yang-Mills theory [Wit92] finds that the corresponding integral over $F = T^*\mathcal{H}$ produces a trivial factor of 1, and this is no longer the case in Chern-Simons theory as is observed in [BW05]. The quantity of interest is then,

$$(136) \quad I(\psi) := \frac{1}{\text{Vol } \mathcal{H}_{\tilde{F}}} \oint_{\tilde{F}} \left[\frac{dY}{2\pi} \right] \exp(t \cdot D\theta),$$

where,

$$\tilde{F} := (\mathfrak{H} \ominus \mathfrak{H}_0) \times F,$$

and,

$$\psi = p + a \in \mathfrak{H}_0,$$

and we set $p = \frac{1}{\epsilon}$ at the end of the computation. Following the exact same reasoning that leads to [BW05, Eq. 5.117], we may identify,

$$(137) \quad I(\psi) = \frac{1}{\text{Vol } \mathcal{H}_0} \frac{e_{\mathcal{H}_0}(\mathcal{M}_P, \mathcal{E}_0)}{e_{\mathcal{H}_0}(\mathcal{M}_P, \mathcal{E}_1)},$$

where $e_{\mathcal{H}_0}(\mathcal{M}_P, \mathcal{E}_0)$, $e_{\mathcal{H}_0}(\mathcal{M}_P, \mathcal{E}_1)$ are the \mathcal{H}_0 -equivariant Euler classes of the bundles associated to \mathcal{E}_0 , \mathcal{E}_1 as in (126), (127) over \mathcal{M}_P . This may be confirmed by direct computation exactly as in [BW05, Appendix D] and we do not repeat this argument here. Define,

$$(138) \quad e(p) := \frac{e_{\mathcal{H}_0}(\mathcal{M}_P, \mathcal{E}_0)}{e_{\mathcal{H}_0}(\mathcal{M}_P, \mathcal{E}_1)}.$$

Then our considerations so far yield,

$$(139) \quad \bar{Z}_{\mathbb{T}}(X, P, k) = \frac{e^{ik \text{CS}_{X,P}(A_P)}}{\text{Vol } I} \int_{\mathcal{M}_P} e(p)|_{\{p=\frac{1}{\epsilon}\}} \exp[\Omega + i\epsilon n \Theta].$$

Note that in deriving (140) that the factor $\text{Vol}(U(1)^2)$ in (135) cancels with a factor in $\text{Vol } \mathcal{H}_0$ in $I(\psi)$. Recall that $\Theta \in H^4(\mathcal{M}_P)$ is the cohomology class corresponding to the degree four element (Y, Y) in the equivariant cohomology $H_{\mathcal{H}}^4(N)$. We observe that in the abelian case $G = \mathbb{T}$ that Θ can also be described in terms of the universal bundle \mathbb{U} ,

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{U} \\ & & \downarrow \\ & & \text{Jac}(\Sigma) \times \Sigma \end{array}.$$

In other words,

$$\Theta = -\frac{1}{2}c_1(\mathbb{U})^2|_{\text{pt.} \in \Sigma},$$

where $\text{Jac}(\Sigma)$ is the Jacobian of Σ . In this case $\Theta = 0$ since the universal bundle \mathbb{U} for \mathbb{T} -bundles is the classical Poincaré line bundle, and the Poincaré line bundle is normalized to have degree $d = 0$ when restricted to the Jacobian of Σ . Thus, our computation produces the simple result,

$$(140) \quad \bar{Z}_{\mathbb{T}}(X, P, k) = \frac{e^{ik \text{CS}_{X,P}(A_P)}}{\text{Vol } I} \int_{\mathcal{M}_P} e(p)|_{\{p=\frac{1}{\epsilon}\}} \exp[\Omega].$$

Our main goal now is to compute/define $e(p)|_{\{p=\frac{1}{\epsilon}\}}$.

7.3. A Final Definition of the Symplectic Abelian Partition Function. We compute/define $e(p)|_{\{p=\frac{1}{\epsilon}\}}$ following the main arguments in [BW05] and adapt their technique to the case of an abelian structure group. The main difference in the case of an abelian structure group shows up in the k -dependence of the partition function $\bar{Z}_{\mathbb{T}}(X, P, k)$. Note that $\epsilon = \frac{2\pi}{k}$. This difference is due to the fact that [BW05] works with irreducible flat connections and the corresponding zeroth cohomology spaces vanish, whereas an abelian structure group necessarily has non-vanishing zeroth cohomology.

Recall,

$$(141) \quad e(p) := \frac{e_{\mathcal{H}_0}(\mathcal{M}_P, \mathcal{E}_0)}{e_{\mathcal{H}_0}(\mathcal{M}_P, \mathcal{E}_1)}.$$

Let $\mathcal{E}_0^{[l]}$, $\mathcal{E}_1^{[l]}$ denote the natural eigenspaces in \mathcal{E}_0 , \mathcal{E}_1 under the action of \mathcal{H}_0 , so that,

$$(142) \quad \mathcal{E}_0^{[l]} = H_{\bar{\partial}}^0(\Sigma, \mathcal{L}^l \otimes \mathfrak{t}),$$

and,

$$(143) \quad \mathcal{E}_1^{[l]} = H_{\bar{\partial}}^1(\Sigma, \mathcal{L}^l \otimes \mathfrak{t}).$$

As in [BW05, Eq. 5.126], we may write,

$$(144) \quad e(p) = \prod_{l \neq 0} \left[\frac{e_{\mathcal{H}_0}(\mathcal{M}_P, \mathcal{E}_0^{[l]})}{e_{\mathcal{H}_0}(\mathcal{M}_P, \mathcal{E}_1^{[l]})} \right] = \prod_{l \geq 1} \left[\frac{e_{\mathcal{H}_0}(\mathcal{M}_P, \mathcal{E}_0^{[l]}) \cdot e_{\mathcal{H}_0}(\mathcal{M}_P, \mathcal{E}_0^{[-l]})}{e_{\mathcal{H}_0}(\mathcal{M}_P, \mathcal{E}_1^{[l]}) \cdot e_{\mathcal{H}_0}(\mathcal{M}_P, \mathcal{E}_1^{[-l]})} \right],$$

where $e_{\mathcal{H}_0}(\mathcal{M}_P, \mathcal{E}_0^{[l]})$, $e_{\mathcal{H}_0}(\mathcal{M}_P, \mathcal{E}_1^{[l]})$ denote the \mathcal{H}_0 equivariant Euler classes of the finite dimensional bundles determined by $\mathcal{E}_0^{[l]}$, $\mathcal{E}_1^{[l]}$ over \mathcal{M}_P . [BW05] then finds a recursive relation between the equivariant Euler classes of $\mathcal{E}_0^{[l]}$, $\mathcal{E}_0^{[-l]}$, $\mathcal{E}_1^{[l]}$, $\mathcal{E}_1^{[-l]}$ by choosing a convenient holomorphic structure on \mathcal{L} . Using this recursive relation [BW05] find,

$$(145) \quad e(p) = \prod_{l \neq 0} \frac{1}{e_{\mathcal{H}_0}(\mathcal{M}_P, T\mathcal{M}_P^l)} [e_{\mathcal{H}_0}(\mathcal{M}_P, \mathcal{V}_P^l)]^{nl},$$

where $T\mathcal{M}_P^l$ denotes the \mathcal{H}_0 equivariant version of the tangent space of \mathcal{M}_P , and \mathcal{V}_P^l denotes a bundle associated to the chosen holomorphic structure and is defined in [BW05, pg. 103]. The result in (145) assumes that points in the moduli space correspond to *irreducible* flat connections so that $\mathcal{E}_0^{[0]}$ vanishes. In our case $\mathcal{E}_0^{[0]} = H_{\bar{\partial}}^0(\Sigma, \mathfrak{t})$ does not vanish and we must revise accordingly. Let \mathcal{I}_P^l denote the \mathcal{H}_0 equivariant bundle over \mathcal{M}_P associated to the bundle with fiber $\mathcal{E}_0^{[0]}$ such that \mathcal{H}_0 acts on the fiber with eigenvalue $-2\pi il$. Our revised version of (145) is then,

$$(146) \quad e(p) = \prod_{l \neq 0} \frac{e_{\mathcal{H}_0}(\mathcal{M}_P, \mathcal{I}_P^l)}{e_{\mathcal{H}_0}(\mathcal{M}_P, T\mathcal{M}_P^l)} [e_{\mathcal{H}_0}(\mathcal{M}_P, \mathcal{V}_P^l)]^{nl}.$$

Following the the exact method leading to [BW05, Eq. 5.144] we may factorize $e(p)$ as a product of the three terms,

$$(147) \quad e(p) = \prod_{l \neq 0} \left[\prod_{j=1}^{\dim \mathbb{T}} (-ilp + \iota_j) \right] \left[\prod_{j=1}^{\dim \mathcal{M}_P/2} \frac{1}{(-ilp + w_j)} \right] \left[\prod_{j=1}^{\dim \mathbb{T}} (-ilp + \nu_j)^{nl} \right],$$

where ι_j, w_j, ν_j are the Chern roots of the bundles $\mathcal{I}_P, T\mathcal{M}_P, \mathcal{V}_P$, respectively. Let,

$$(148) \quad f_{\mathcal{I}}(z) = \prod_{l \neq 0} \prod_{j=1}^{\dim \mathbb{T}} (-il + z\iota_j),$$

$$(149) \quad f_{\mathcal{M}}(z) = \prod_{l \neq 0} \prod_{j=1}^{\dim \mathcal{M}_P/2} (-il + zw_j)^{-1},$$

$$(150) \quad f_{\mathcal{V}}(z) = \prod_{l \neq 0} \prod_{j=1}^{\dim \mathbb{T}} (-il + z\nu_j)^{nl},$$

where $z = 1/p$ is a formal parameter. Factoring out p from each of the three terms in (147) and using the Riemann zeta function to define the infinite products,

$$\begin{aligned} \prod_{l \geq 1} p^{2 \dim \mathbb{T}} &= \exp(2 \dim \mathbb{T} \cdot \ln p \cdot \zeta(0)) = p^{-\dim \mathbb{T}}, \\ \prod_{l \geq 1} p^{-\dim \mathcal{M}_P} &= \exp(-\dim \mathcal{M}_P \cdot \ln p \cdot \zeta(0)) = p^{\dim \mathcal{M}_P/2}, \end{aligned}$$

we may write,

$$(151) \quad e(p) = p^{\frac{1}{2}(\dim \mathcal{M}_P - 2 \dim \mathbb{T})} \cdot f_{\mathcal{I}}(z) \cdot f_{\mathcal{M}}(z) \cdot f_{\mathcal{V}}(z).$$

One may apply exactly the same reasoning that leads to [BW05, 5.167] using appropriate zeta and eta function regularizations to rigorously define the quantities $f_{\mathcal{I}}(z), f_{\mathcal{M}}(z), f_{\mathcal{V}}(z)$. We will not repeat the full argument here and instead we point out the the main differences that arise for the case of an abelian group \mathbb{T} . First, since $\mathcal{M}_P \simeq \mathbb{T}^{2g}$, we have,

$$\begin{aligned} c(\mathcal{M}_P) &:= c(T\mathcal{M}_P), \\ &= \prod_{j=1}^{\dim \mathcal{M}_P/2} c(L_j) = \prod_{j=1}^{\dim \mathcal{M}_P/2} (1 + x_j), \end{aligned}$$

where $L_j = T\Sigma_j$, $x_j = c_1(L_j) \in H^2(\Sigma_j, \mathbb{Z})$, and $\Sigma_j \simeq (\mathrm{U}(1))^2$. Since Σ_j are Lie groups, the tangent bundles $T\Sigma_j$ are trivial and hence,

$$x_j = c_1(T\Sigma_j) = 0.$$

Thus,

$$(152) \quad \hat{A}(\mathcal{M}_P) = \prod_{j=1}^{\dim \mathcal{M}_P/2} \frac{x_j/2}{\sinh(x_j/2)} = 1.$$

Clearly, $c_1(T\mathcal{M}_P) = 0$ as well. Using these observations and the method leading to [BW05, 5.167], we have,

$$(153) \quad f_{\mathcal{I}}(p) = (2\pi)^{\dim \mathbb{T}}, \quad f_{\mathcal{M}}(p) = \left(\frac{1}{2\pi}\right)^{\dim \mathcal{M}_P/2}, \quad f_{\mathcal{V}}(p) = \exp\left(\frac{-i\pi}{2}\eta_0\right),$$

where η_0 is the adiabatic eta invariant of X and was first defined in [Nic00]. Note that η_0 is computed explicitly in [Bea07, Appendix C] and is given by,

$$\eta_0 = N \left(\frac{c_1(X)}{6} - 2 \sum_{j=1}^M s(\alpha_j, \beta_j) \right),$$

where $s(\alpha, \beta) := \frac{1}{4\alpha} \sum_{j=1}^{\alpha-1} \cot\left(\frac{\pi j}{\alpha}\right) \cot\left(\frac{\pi j\beta}{\alpha}\right) \in \mathbb{Q}$ is the classical Rademacher-Dedekind sum and $[g, n; (\alpha_1, \beta_1), \dots, (\alpha_M, \beta_M)]$ (for $\gcd(\alpha_j, \beta_j) = 1$) are the Seifert invariants of X . Overall, we have,

$$(154) \quad e(p) = \left(\frac{1}{2\pi} \right)^{\dim H^1(X, \mathfrak{t})} \cdot k^{\frac{1}{2}(\dim H^1(X, \mathfrak{t}) - 2 \dim H^0(X, \mathfrak{t}))} \cdot \exp\left(-\frac{i\pi}{2}\eta_0\right),$$

where we have used the fact that $\dim H^1(X, \mathfrak{t}) = \dim \mathcal{M}_P$, $\dim H^0(X, \mathfrak{t}) = \dim \mathbb{T}$ and $p = \frac{1}{\epsilon} = \frac{k}{2\pi}$. Finally, as is noted in [BW05, Pg. 89-92] the derivation of the partition function is done implicitly with respect to a choice of the so called *Seifert framing* on X . This choice of framing results in a difference of a factor of $e^{i\delta\Psi}$ in the partition function relative to the canonical framing, where for general gauge group, [BW05, Eq. 5.101],

$$(155) \quad e^{i\delta\Psi} = \exp\left(\frac{i\pi\Delta_G}{4} - \frac{i\pi\Delta_G\check{c}_g}{12(k+\check{c}_g)}\theta_0 + \frac{i\pi}{2}\eta_0\right).$$

For the case of gauge group \mathbb{T} we take $\Delta_G = N$ and $\frac{i\pi\Delta_G\check{c}_g}{12(k+\check{c}_g)}\theta_0$ to vanish. Thus, we should replace,

$$(156) \quad e^{(-\frac{i\pi}{2}\eta_0)},$$

in the partition function with,

$$(157) \quad e^{(\frac{i\pi N}{4} - \frac{i\pi}{2}\eta_0)}.$$

Before we make our rigorous definition, we recall the following,

- $m_X := \frac{N}{2}(\dim H^1(X; \mathbb{R}) - 2 \dim H^0(X; \mathbb{R}))$,
- A_P flat connection on principal $U(1)$ -bundle P with Chern-Simons invariant $\text{CS}_{X,P}(A_P)$,
- $[g, n; (\alpha_1, \beta_1), \dots, (\alpha_M, \beta_M)]$ (for $\gcd(\alpha_j, \beta_j) = 1$) are the Seifert invariants of X ,
- $\eta_0 = N \left(\frac{c_1(X)}{6} - 2 \sum_{j=1}^M s(\alpha_j, \beta_j) \right)$ is the *adiabatic eta invariant* of the Seifert manifold X introduced in [Nic00],
- $s(\alpha, \beta) := \frac{1}{4\alpha} \sum_{j=1}^{\alpha-1} \cot\left(\frac{\pi j}{\alpha}\right) \cot\left(\frac{\pi j\beta}{\alpha}\right) \in \mathbb{Q}$ is the classical Rademacher-Dedekind sum,
- $\mathcal{M}_P \simeq H^1(X, \mathfrak{t})/H^1(X, \Lambda) \simeq \mathbb{T}^{2g}$ moduli space of flat \mathbb{T} connections on P .

Plugging the result for $e(p)$ in (154) into (140) we make the following,

Definition 23. Let $k \in \mathbb{Z}$, and let X be a closed oriented three-manifold that admits a quasi-regular Sasakian structure (κ, Φ, ξ, g) , with associated principal bundle structure,

$$\begin{array}{ccc} \mathbb{U}(1) & \hookrightarrow & X \\ & \downarrow & \\ & \Sigma & \end{array}$$

Define the symplectic abelian Chern-Simons partition function,

$$(158) \quad \bar{Z}_{\mathbb{T}}(X, k) = \sum_{[P] \in \text{Tors } H^2(X, \Lambda)} \bar{Z}_{\mathbb{T}}(X, P, k),$$

where,

$$(159) \quad \bar{Z}_{\mathbb{T}}(X, P, k) = k^{m_X} e^{ik \text{CS}_{X,P}(A_P)} e^{i\pi(\frac{N}{4} - \frac{1}{2}\eta_0)} \int_{\mathcal{M}_P} K_X \cdot \omega_P,$$

where, $m_X := \frac{N}{2}(\dim H^1(X; \mathbb{R}) - 2 \dim H^0(X; \mathbb{R}))$, $K_X := \frac{1}{|c_1(X) \cdot \prod_i \alpha_i|^{N/2}}$, $\omega_P := \frac{(\sum_{j=1}^g d\theta_j \wedge d\bar{\theta}_j)^{gN}}{(gN)!(2\pi)^{2gN}}$.

Remark 24. Note that we assumed that X was a principal $\mathbb{U}(1)$ bundle over a smooth surface Σ in the above derivation of the partition function. We accordingly revise the factor $1/\text{Vol}(I) = 1/|c_1(X)|$ in the partition function to K_X in order to restore topological invariance in the case when orbifold points are present.

8. THE MODULI SPACE OF ABELIAN CONNECTIONS

In order to be more explicit in our computation of the abelian Chern-Simons partition function we study the moduli space of flat connections, \mathcal{M}_X , over a Sasakian manifold X more closely in this section. One of the virtues of the abelian theory is that the moduli space of flat connections is relatively simple. Let \mathbb{T} denote a compact, connected abelian Lie group of dimension N , \mathfrak{t} denote its Lie algebra and $\Lambda \subset \mathfrak{t}$ the integral lattice. Let $\text{Tors } H^2(X, \Lambda)$ denote the torsion subgroup of $H^2(X, \Lambda)$. In general, we have the following,

Proposition 25. [Man98] *Let X be a smooth three manifold and let \mathcal{M}_X be the moduli space of flat abelian \mathbb{T} connections on X . Then,*

- *There is a natural identification,*

$$\mathcal{M}_X = H^1(X, \mathbb{T}),$$

and,

- $|\pi_0(\mathcal{M}_X)| = |\text{Tors } H^2(X, \Lambda)|$ and each connected component of \mathcal{M}_X is diffeomorphic to the torus $H^1(X, \mathfrak{t})/H^1(X, \Lambda)$.

Our goal in this section is to obtain an expression for the number of components of \mathcal{M}_X in terms of Seifert data for a given Sasakian three manifold X . The main result of this section is contained in Prop. (27) below. Note that a Sasakian three manifold X admits a natural Seifert structure. For a general overview of the relevant geometric background see [BG08]. For us, this will mean that X admits a locally free $\mathbb{U}(1)$ action and can be written as a *non-trivial* $\mathbb{U}(1)$ -bundle over an orbifold,

$$\begin{array}{ccc} \mathbb{U}(1) & \hookrightarrow & X \\ & \downarrow & \\ & \Sigma & \end{array}$$

where $\Sigma = \{|\Sigma|, \mathcal{U}\}$ is an orbifold with underlying space $|\Sigma|$ a surface of genus g .

Remark 26. *The key property of Sasakian manifolds for us in this section is that they are precisely the three manifolds with non-vanishing first orbifold Chern-class, $c_1(X)$ [Sco83].*

Recall that the fundamental group of a Seifert manifold X , $\pi_1(X)$, is generated by the following elements, [Orl72],

$$\begin{aligned} a_p, b_p, \quad p = 1, \dots, g, \\ c_j, \quad j = 1, \dots, M, \\ h, \end{aligned}$$

which satisfy the relations,

$$\begin{aligned} [a_p, h] &= [b_p, h] = [c_j, h] = 1, \\ c_j^{\alpha_j} h^{\beta_j} &= 1, \\ \prod_{p=1}^g [a_p, b_p] \prod_{j=1}^M c_j &= h^n. \end{aligned}$$

The generator h is associated to the generic $\mathbb{U}(1)$ fiber over Σ , the generators a_p, b_p come from the $2g$ non-contractible cycles on Σ , and the generators c_j come from the small one cycles in Σ around each of the orbifold points p_j . The moduli space of flat connections over X can be realized as the space of homomorphisms from $\pi_1(X)$ to \mathbb{T} . Consider,

$$\rho : \pi_1(X) \rightarrow \mathbb{T},$$

and let, $C_j = \rho(c_j), B = \rho(h)$. Since \mathbb{T} is abelian, the generating relations for $\pi_1(X)$ translate into the following restrictions on ρ ,

$$\begin{aligned} C_j^{\alpha_j} B^{\beta_j} &= 1, \quad j = 1, \dots, M, \\ \prod_{j=1}^M C_j \cdot B^{-n} &= 1, \end{aligned}$$

where,

$$(160) \quad C_j = \left[e^{2\pi i \psi_j^1}, \dots, e^{2\pi i \psi_j^k}, \dots, e^{2\pi i \psi_j^N} \right] \in \mathbb{T}, \quad j = 1, \dots, M,$$

$$(161) \quad B = \left[e^{2\pi i \psi_0^1}, \dots, e^{2\pi i \psi_0^k}, \dots, e^{2\pi i \psi_0^N} \right] \in \mathbb{T},$$

for some $\psi_j^k \in [0, 1)$, $j = 0, \dots, M$ and $k = 1, \dots, N$. Let $K_{l,j}$ be the following $(M+1) \times (M+1)$ matrix,

$$(162) \quad K = \begin{pmatrix} -n & 1 & 1 & \cdots & 1 \\ \beta_1 & \alpha_1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ \beta_{M-1} & \vdots & 0 & \alpha_{M-1} & 0 \\ \beta_M & 0 & \cdots & 0 & \alpha_M \end{pmatrix}$$

We condense the above restrictions and write,

$$(163) \quad \prod_{j=0}^M e^{2\pi i K_{l,j} \psi_j^k} = 1,$$

for each $k = 1, \dots, N$. For simplicity, let us assume that $N = 1$ in the remainder. Let $\mathbf{v} = (\psi_0, \psi_1, \dots, \psi_M) \in \mathbb{R}^{M+1}$. Then \mathbf{v} solves the above equation, (163), if and only if,

$$(164) \quad K \cdot \mathbf{v} = \mathbf{w} \in \mathbb{Z}^{M+1}.$$

Let $\mathbf{d} := |\det K| \in \mathbb{Z}$. Observe that $|\mathbf{d}| = |\det K| = \left| \left(\prod_{j=1}^M \alpha_j \right) \cdot \left(n + \sum_{j=1}^M \frac{\beta_j}{\alpha_j} \right) \right| = \left| \left(-\prod_{j=1}^M \alpha_j \right) \cdot c_1(X) \right| \neq 0$, since $\alpha_j \neq 0, \forall 1 \leq j \leq M$, and the orbifold first Chern number $c_1(X) \neq 0$ by assumption (i.e. the Seifert structure on X is associated to a Sasakian

structure). Thus, K is invertible and,

$$\mathbf{v} = K^{-1}\mathbf{w} = \frac{(\text{Cof } K)^T}{\mathbf{d}}\mathbf{w},$$

where the cofactor matrix $\text{Cof } K \in M_{M+1}(\mathbb{Z})$. We conclude that \mathbf{v} must necessarily satisfy,

$$\mathbf{v} \in \frac{1}{\mathbf{d}}\mathbb{Z}^{M+1},$$

if it is a solution of (163). Since solutions of (163) are defined only up to the integers, we may therefore take $\mathbf{v} \in \frac{1}{\mathbf{d}}\mathbb{Z}_{\mathbf{d}}^{M+1}$, where $\mathbb{Z}_{\mathbf{d}} := \mathbb{Z}/(\mathbf{d} \cdot \mathbb{Z})$, the set of integers modulo \mathbf{d} . We then have a one to one correspondence between solutions $\mathbf{v} \in \frac{1}{\mathbf{d}}\mathbb{Z}_{\mathbf{d}}^{M+1}$ and vectors $\tilde{\mathbf{v}} = \mathbf{d}\mathbf{v} \in \mathbb{Z}_{\mathbf{d}}^{M+1}$ such that $K\tilde{\mathbf{v}} = 0 \in \mathbb{Z}_{\mathbf{d}}^{M+1}$. Viewing,

$$K : \mathbb{Z}_{\mathbf{d}}^{M+1} \rightarrow \mathbb{Z}_{\mathbf{d}}^{M+1},$$

as a $\mathbb{Z}_{\mathbf{d}}$ -module homomorphism, then solutions of (163) are given exactly by $\frac{1}{\mathbf{d}} \cdot \text{Ker}(K)$. Observe that the last M columns of K form an independent set and therefore $\text{rank}(K) \geq M$. One can show that in fact $\text{rank}(K) = M$. Let,

$$\hat{\alpha}_i := \frac{1}{\alpha_i} \prod_{j=1}^M \alpha_j.$$

Then it is not hard to show that,

$$\mathbf{s} := \left[-\prod_{j=1}^M \alpha_j, \hat{\alpha}_1\beta_1, \dots, \hat{\alpha}_M\beta_M \right] \in \text{Ker}(K) \subset \mathbb{Z}_{\mathbf{d}}^{M+1},$$

and therefore the kernel is cyclicly generated by $\mathbf{s} \neq 0$,

$$\text{Ker}(K) = \langle \mathbf{s} \rangle \simeq \mathbb{Z}_{\mathbf{d}}.$$

Since,

$$\mathcal{M}_X \simeq \mathbb{T}^{2g} \times \text{Tors}(H^2(X, \Lambda)) \simeq \text{Hom}(\pi_1(X), \mathbb{T}),$$

by Prop. (25) in general, we therefore obtain the following,

Theorem 27. *Given a closed, oriented Seifert three manifold X such that $c_1(X) \neq 0$ (i.e. a Sasakian three manifold) then,*

$$\mathcal{M}_X \simeq \mathbb{T}^{2g} \times \text{Tors}(H^2(X, \Lambda)) \simeq \text{Hom}(\pi_1(X), \mathbb{T}),$$

where, $|\text{Tors } H^2(X, \Lambda)| = \mathbf{d}^N = |c_1(X) \cdot \prod_{j=1}^M \alpha_j|^N$.

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